Compute: follow a fixed procedure and produce an answer (halt), aka, **algorithm**.

What can be computed? What cannot? What can be efficiently computed (and how)?

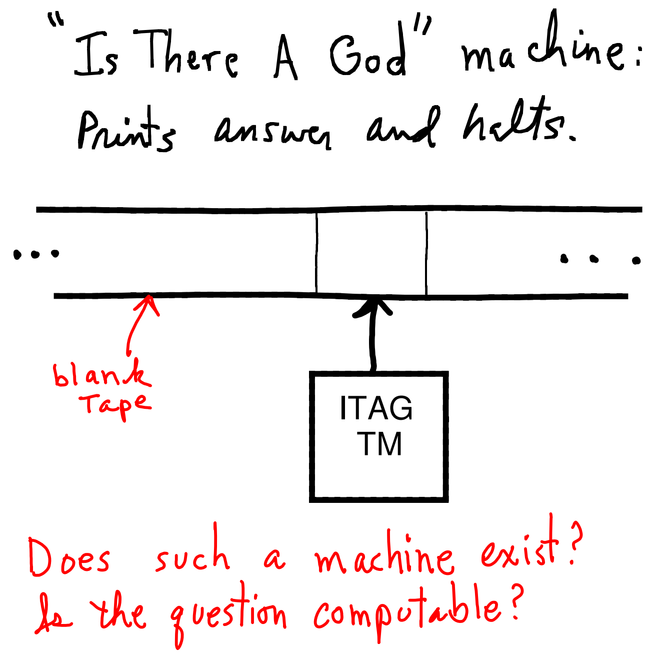
If a single question **really is answerable "yes" or "no"**, then one of the machines, M_yes or M_no, computes the answer. We might not know which one is correct.

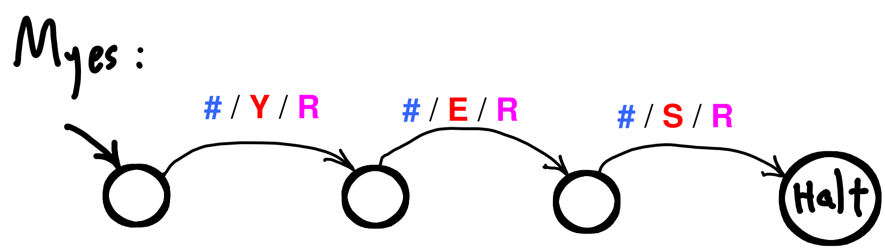
Any finite set of examples can be computed: just **make a table and look up the answer**. Just because you don't know how doesn't mean it can't be done.

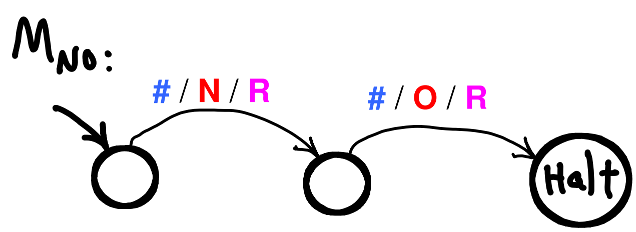
Are all programs algorithms? **No**.

```
for (i = 1; i > 0; i = 1) {
    j = j+1;
}
```

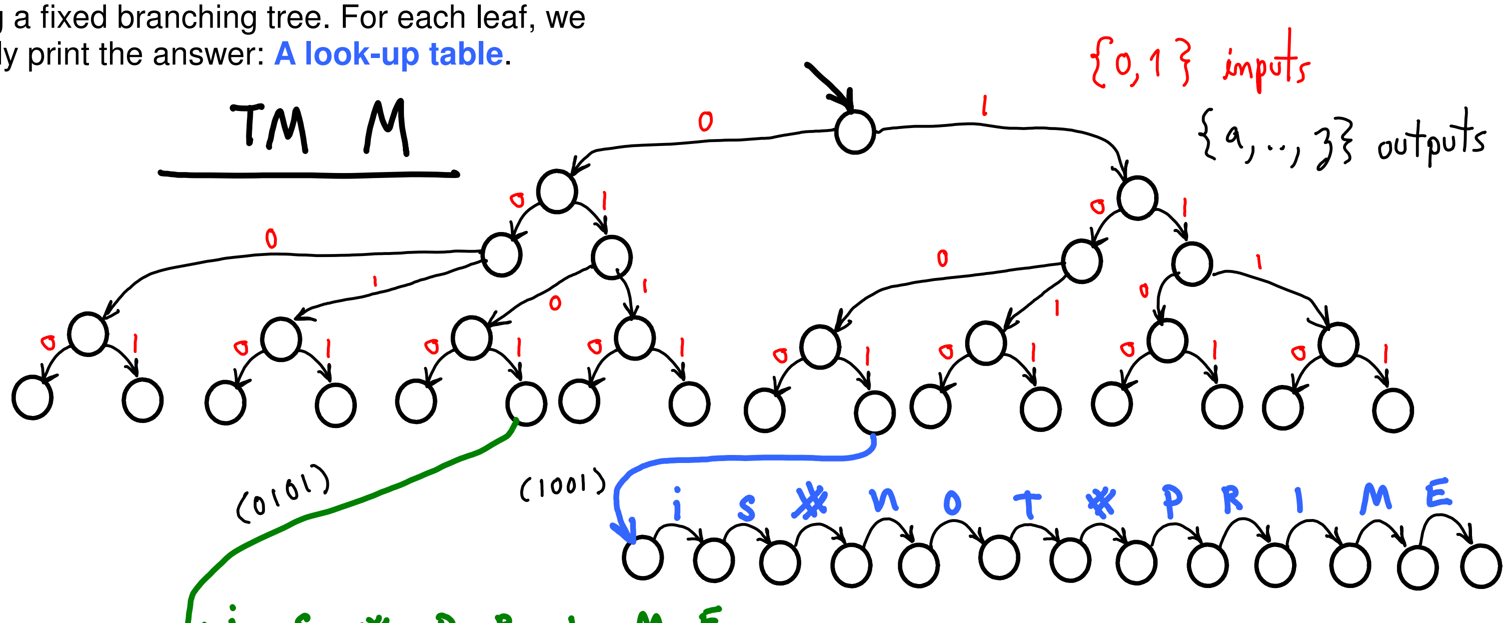
Q. Are all TMs algorithms?







We can **decode any finite set of questions** using a fixed branching tree. For each leaf, we simply print the answer: **A look-up table**.



M is an algorithm solving Primality test for the numbers { 0, 1, ... , 15 }.

Computability (aka, recursive)

Fermat's Last Theorem

There are no solutions to,

$$x^n + y^n = z^n$$

where $n, x, y,$ *and* z are positive integers and $n > 2$.

(Proved in 1995: Frey, Ribet, Wiles, and Taylor.)

Suppose we didn't know whether it was true or not.

Suppose we asked, Is the question,

"Is Fermat's Last Theorem true?" a computable question?

$\Rightarrow$ Of course. Use M_{yes} or M_{no}.

Supposed we asked if this is computable?

Given some positive integer $n > 2$, is there a solution to,

$$x^n + y^n = z^n$$

where $x, y,$ *and* z are positive integers?

If Fermat's Last Theorem is true, then

M_{no} will work w/o modification. $\Rightarrow$ is computable.

Suppose it weren't true? That is, there are sol'ns for some n, but not all n.

How would we go about computing the answer?

```
FLT( n )
   pick next ( x, y, z )
   check whether  x^n + y^n == z^n
   if yes, print "there is a sol'n" ,  halt;
   else, repeat
```

This works when there is a sol'n for a particular n.

?!? Will it halt? For every n?

Diagonalization

How many questions are there? How many TMs?

In our encoding, we used a string of 0s and 1s to represent a TM. Symbol set is {0, 1}.

--- **Each TM** can be identified with **an integer**. (There are infinitely many machines that do the same thing.)

--- **Each input tape** configuration can be identified with **an integer**.

--- **Each output tape** configuration can be identified with **an integer**.

--- A **TM** can be looked at as **an integer function**: given input, **x**, machine **M** produces integer **M(x)**.
---(**M** might loop forever on some inputs; if so, then **M** is a "partial" function.)

List all TM functions

unary encoding of a TM is a binary number:

10010010110110101100101010101110011011011010100110101101101110000

prepend a 1

input

an integer, encoded input tape: ...000...010 is 2

	0	1	2	3	4 ...
M_0 :	$M_0(0)$	$M_0(1)$	$M_0(2)$	$M_0(3)$	...
TM M_1 :	$M_1(0)$	$M_1(1)$	$M_1(2)$	∞	...
M_2 :	$M_2(0)$	$M_2(1)$	...		
M_3 :	...				

← all outputs for $M_\emptyset$

← all outputs for M_1

M_1 loops forever when started w/ tape configuration 3.

an integer, encoded description of M_3

an integer, encoded tape when M_2 halts on input 1.

Q. Can you **encode** an **arbitrary input tape**, in a **arbitrary symbol set**, using **only {0, 1}?** Hint, use unary encoding.

(Recall, only a finite portion of tape is non-blank.)
That binary string is an integer.

Computable (real) numbers:
Given e, output finite number of digits of x so that the output is within e of x.
π is such a number.

Q. Are there incomputable reals?

How many integer functions are there?

Consider a function $g(\,)$, which we describe by saying that $g(\,)$ **is different** from **all the functions** in the list above. How? Because $g(\,)$ is,

not the same as M_0 for the **first output, g(0)**,
and it is,
not the same as M_1 for the **second output, g(1)**,
and it is,
...

--- **Diagonalization:**

$g(0)$ **!= $M_0(0)$**
$g(1)$ **!= $M_1(1)$**
$g(2)$ **!= $M_2(2)$**
... (forever)

---- $g(\,)$ is **different from every function** in the list; so,
$g(\,)$ is **not in the list**!

--- How many different ways are there to pick g()?
$g(0)$ is any element from N - { $M_0(0)$ }
$g(1)$ is any element from N - { $M_1(1)$ }
$g(2)$ is any element from N - { $M_2(2)$ }
...

There are so many different functions, ***g( ),*** proportionally, that the **probability of randomly** picking a function from **a bag of integer functions** and having that **function** correspond to **some TM is 0**.

[What the heck does that really mean?]

That is,

There are a lot of functions (more than all the positive integers).

Nearly all are incomputable

for every input, choose g's output

input:	0	1	2	...
M_0	$\neq$	$M_0(1)$	$M_0(2)$	...
M_1	$M_1(0)$	$\neq$	$M_1(2)$	...
M_2	$M_2(0)$	$M_2(1)$	$\neq$	...
	...			⋱

Is g the same as some TM M_i?
Suppose it is the same as M_k. Look at the k^{th} row,
For input k, $g(k) \neq M_k(k)$,
Oops.

Maybe it only means we don't know how to arrange an infinite list of TMs? We are limited in our own computing power?

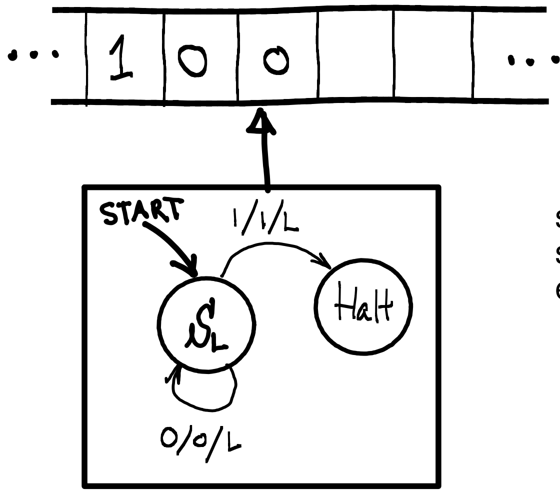
How "numerous" is "infinity to the infinity"?

How can we know we are able to produce g() this way?

As long as we are building TMs, lets see how to simplify our work.
How about combining two TMs to make a new one?

M_1

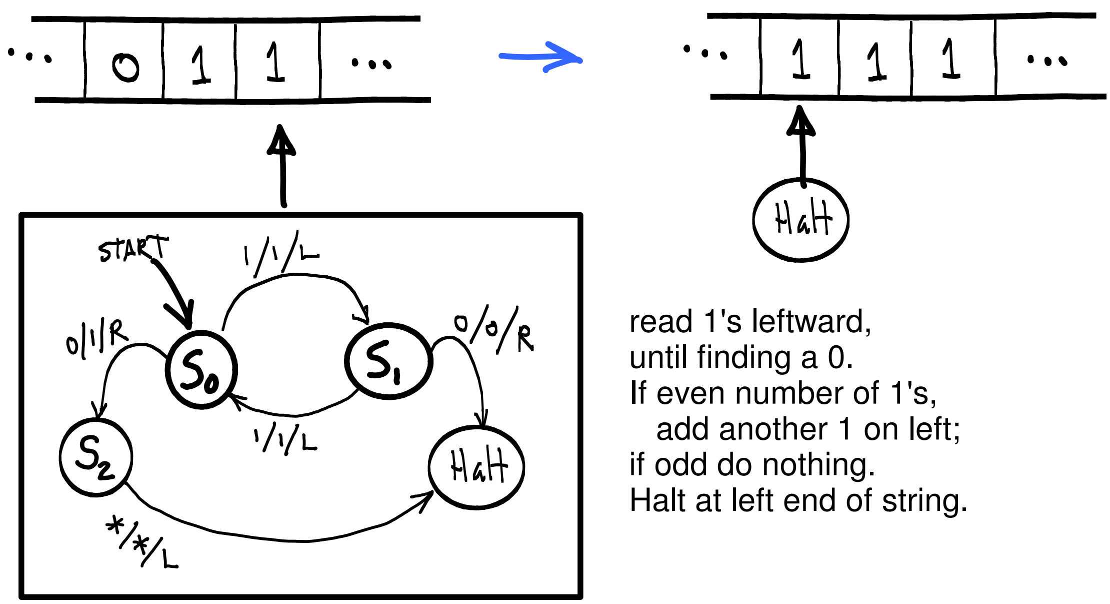
"spin left 0's"



skip 0's to the left,
stop at the first 1,
end up w/ R/W to its left.

M_2

"odd Parity fix"



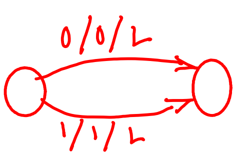
read 1's leftward,
until finding a 0.
If even number of 1's,
add another 1 on left;
if odd do nothing.
Halt at left end of string.

new notation

* means, "any symbol"

//L means, for any symbol *, write that symbol, move L.

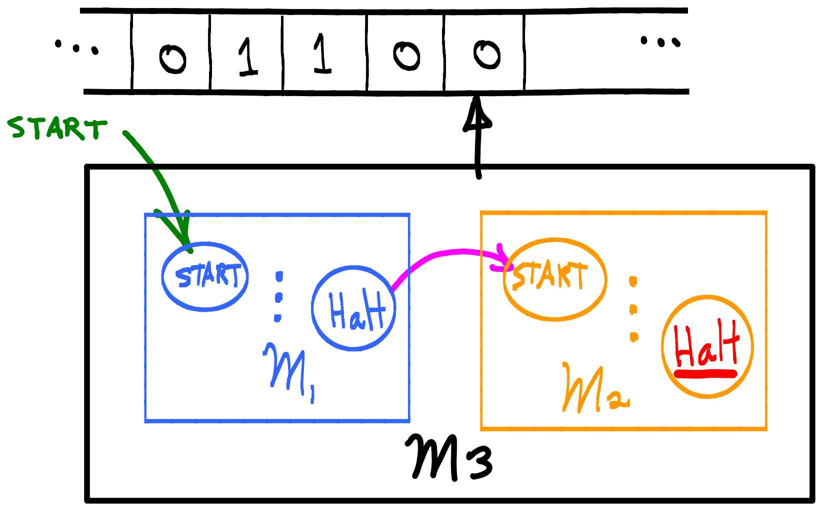
Shorthand for, e.g.,



$\Sigma = \{0, 1\}$

M3

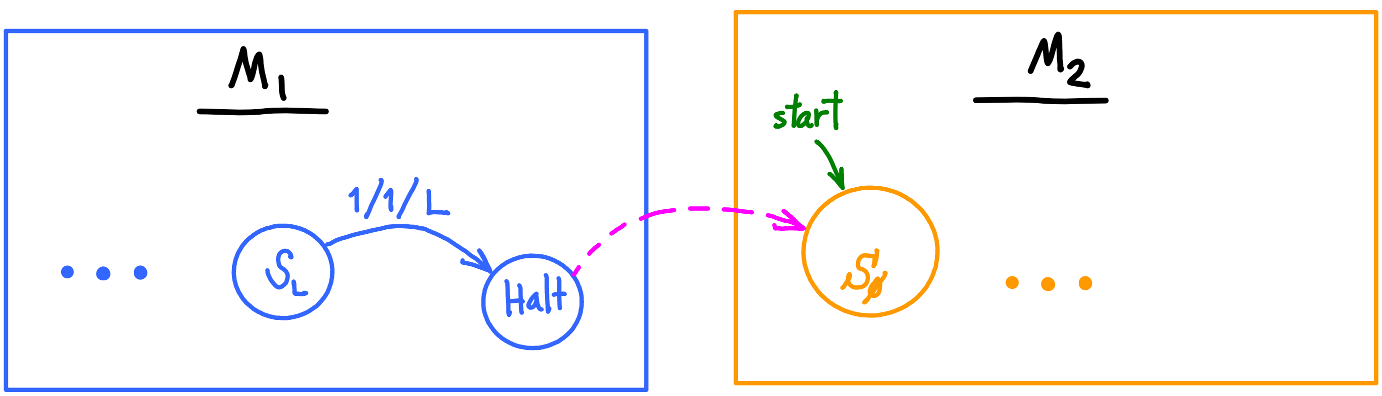
"Spin-left ϕ_s"
Then
"parity fix"



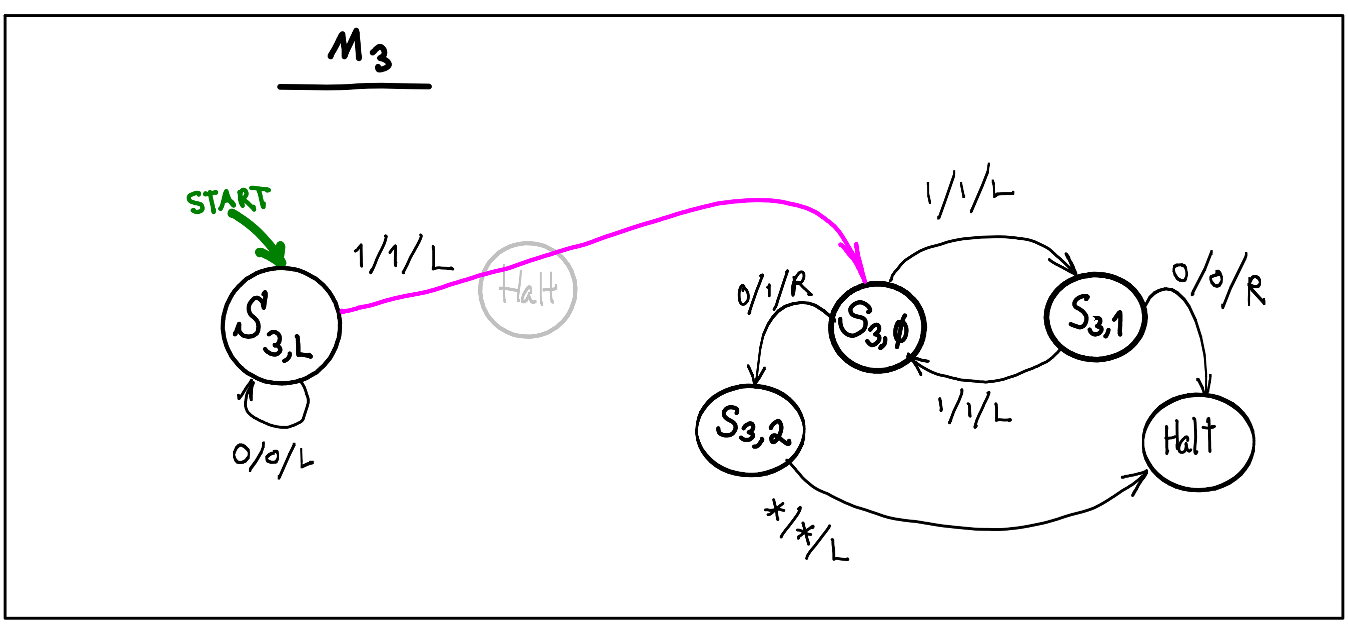
M3 starts in M1's start state.

- Every M1 state transition that goes to **M1**'s "**HALT**" state is instead connected to **M2**'s **START** state.

- **M3**'s halting state is **M2**'s "**HALT**" state.



Means



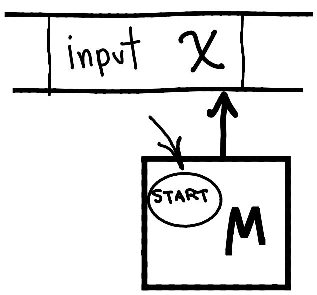
Lemma:
All TM's with x as input, either (1) HALT or (2) LOOP FOREVER. (exercise: prove the lemma.)

A very special integer function: The **Halting function**:

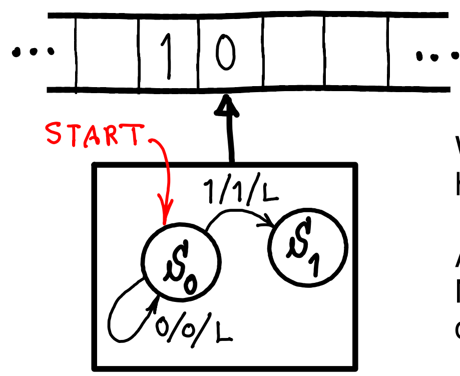
input: integer **xM**	(**xM** == an **encoding of input x followed by an encoding of M**.)
output: "**1**" if **xM** would HALT; "**0**" otherwise.	(**xM** == **M reading x as its input**.)

Question:
Would xM eventually
halt,
or
loop forever?

We do not run M on input x to see if it will halt. It is a mathematical fact that it would either halt or loop. Halting (N) is that fact.



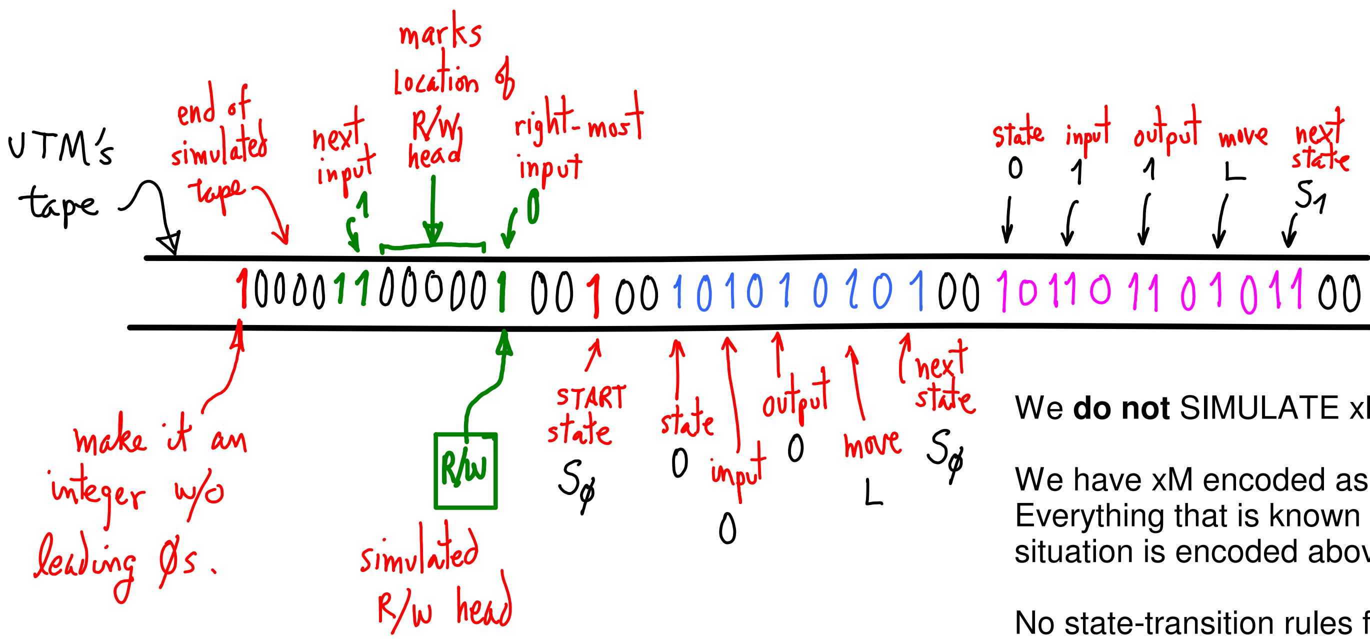
for example:



Will this machine with this input halt or loop forever?

Assume:
No transitions out from a state defines it as a halting state.

The above can **encoded as a single integer**. Given a UTM, we would simulate this situation by putting on the UTM's input tape: An **encoded input tape** containing **x**, an **encoded R/W head location**, an **encoded current state** (start-state initially), and an **encoded rule table.** Put a **1 on the left**, and the **encoded tape represents a non-zero integer**:

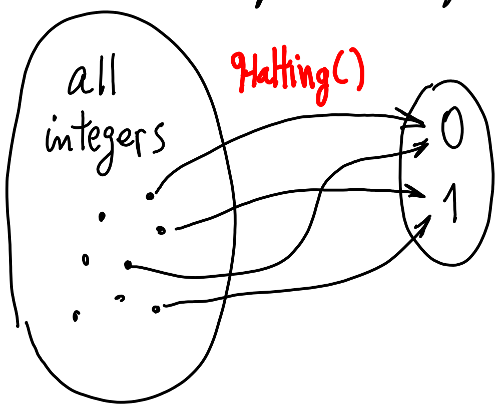


We **do not** SIMULATE xM.

We have xM encoded as an integer. Everything that is known about the situation is encoded above.

No state-transition rules for State-1 make it a halting state.

Does the integer function Halting(N) exist?
Is it a function from integers to integers?

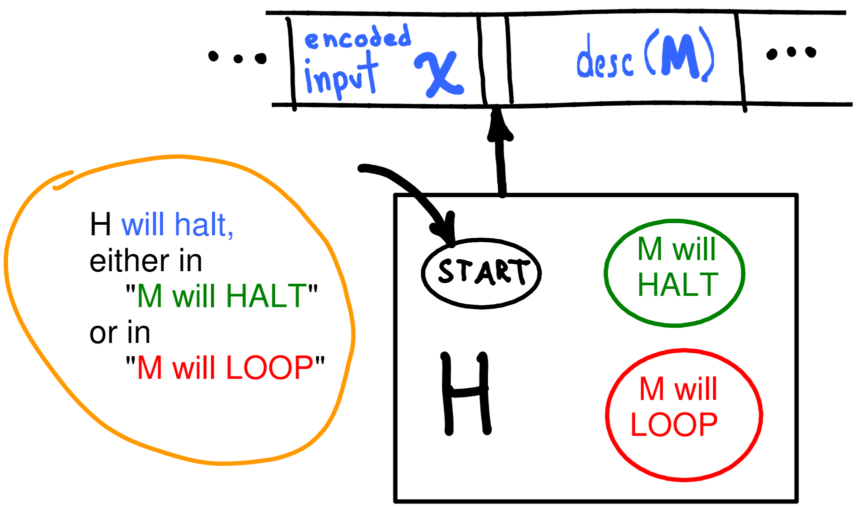


Most N are not legal encodings of xM for some TM M and input x.

How does Halting() map in those cases? We don't care. (Make it 0.)

Is there a mapping for every xM?

For every N?



Is there a Turing Machine,

H,

that computes the function,

Halting(N)?

It only has to be "correct" for legal

N = xM

For N not a legal xM encoding, we don't care which state it halts in

We can assume H will:

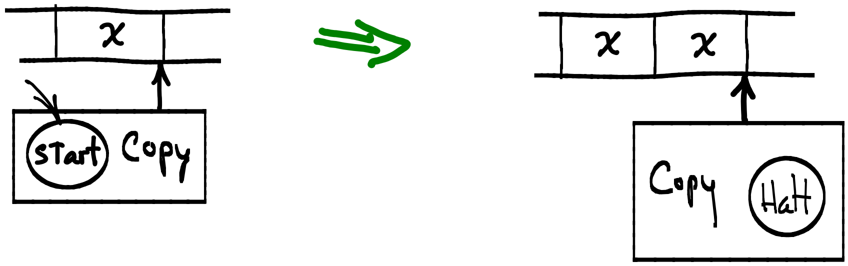
output a 0
when it transitions to the state
"M will HALT";

output a 1
when it transitions to the state
"M will LOOP".

Q. Can there be H, a TM that computes this function. Is it possible?
Asummption: Either (**H exists**) IS TRUE, or (**H does not exist**) IS TRUE.

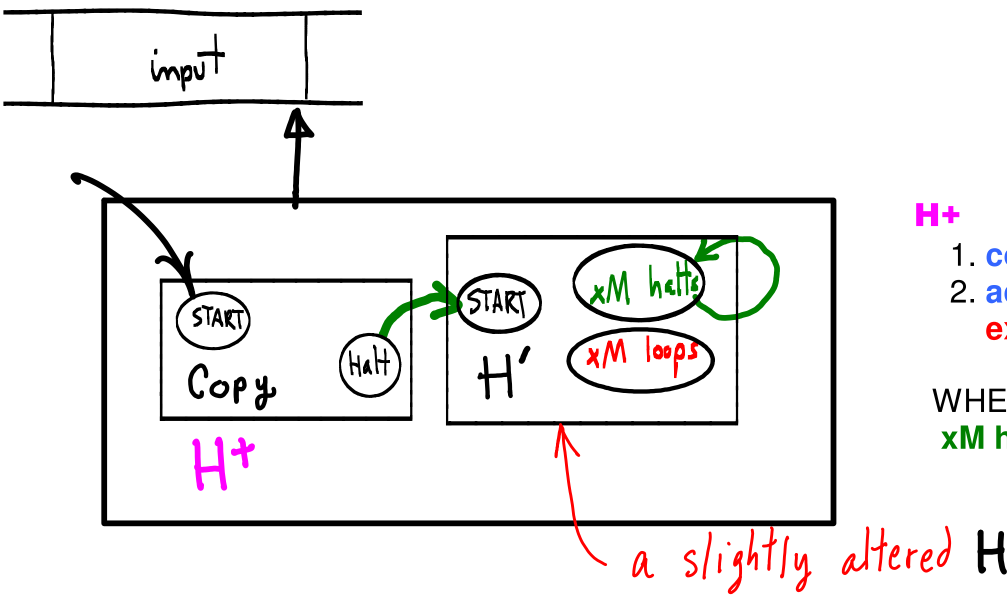
Suppose (**H exists**) **IS TRUE**.

Then we can **build another machine**, **H+**, using **H** and a "**Copy**" TM.



"Copy" TM

Halts after copying its input.



H+

1. **copies its input**.
2. **acts as H would**,
   except:

WHEN **H+** reaches
xM halts, **H+ LOOPS**.

Aside, altering H to create H'

Because we assumed there is a TM, H, then,

there must be a rule table for H.

Consider the rules for H's state labeled "halts" and "loops".

These are both halting states that cause H to stop operating.

===> For H,

There are no rules for the states "halts" and "loops":

E.g., in contradiction, suppose this was a rule for H:

[state="halts", symbol="0", output="1", move="L", nextState="halts"]

Then, H would not halt in state "halts", i.e., **"halts" would not be a halting state.**

We can make a new TM, H':

1. **make a copy of all H's rules**

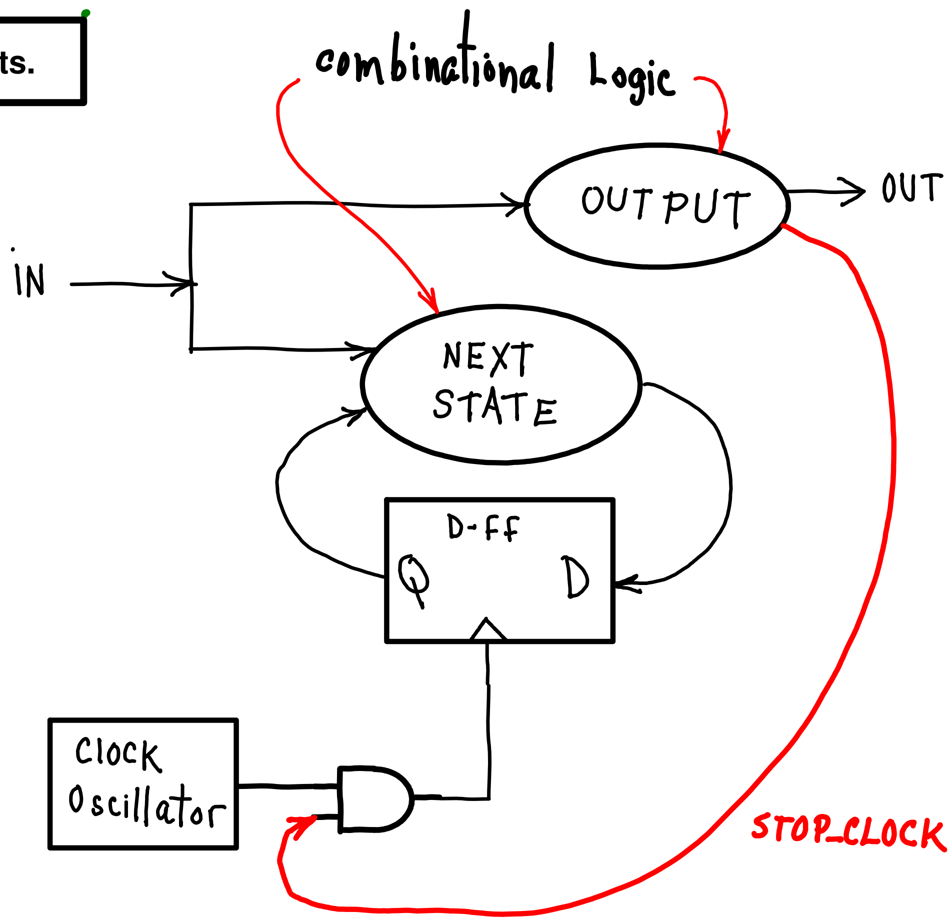
2. **Add these rules:**

[state="halts", symbol="0", output="0", move="L", nextState="halts"]
[state="halts", symbol="1", output="1", move="L", nextState="halts"]
[state="halts", symbol=" " , output=" " , move="L", nextState="halts"]

If the new machine, H', ever reaches its "halts" state, it will,

loop forever, always going back to its "halts" state.

Aside, How To Build A TM That Halts.



To Implement a TM, we build its

Finite State Machine

The input "IN" comes from reading the tape.

The output "OUT" is written to the tape. (Assume it also controls moving the R/W head.)

The "CLOCK Oscillator" produces the clock signal.

The output "STOP_CLOCK" is always 1, execept for any halting state, then it is 0.

Once STOP_CLOCK == 0

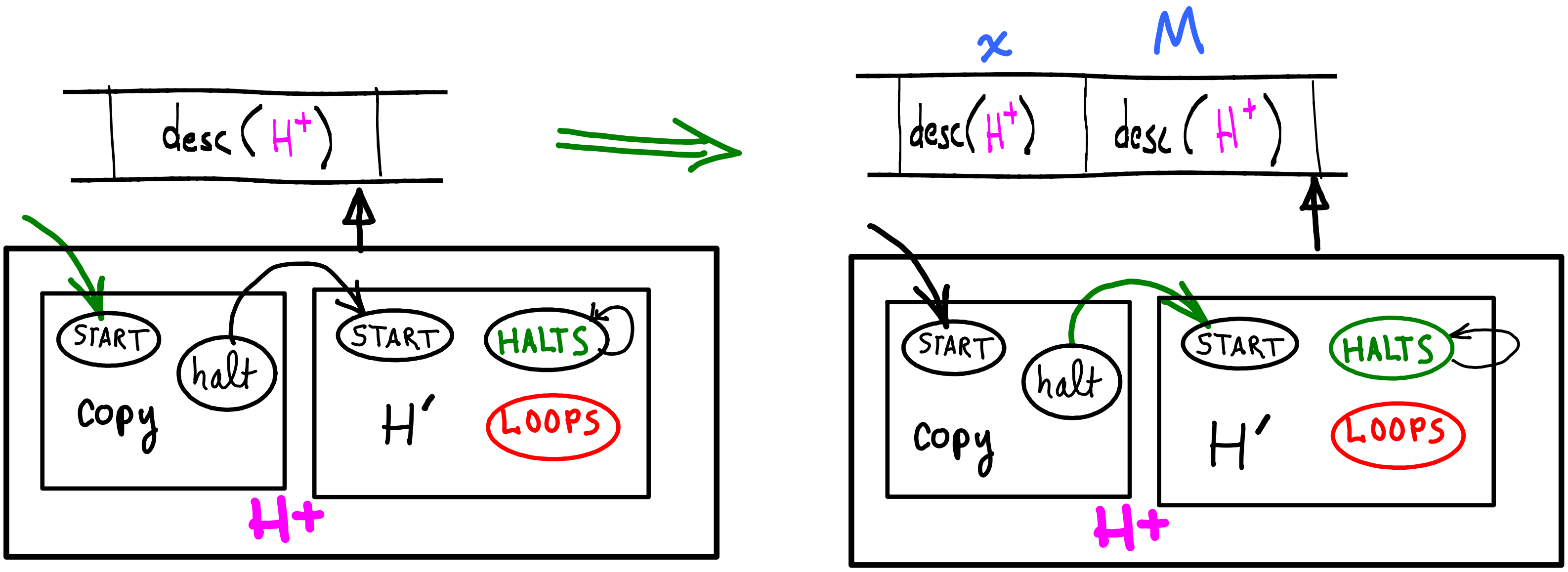
the machine cannot change state;

All outputs are 0 for that state:

nothing happens.

We could have that STOP_CLOCK signal turn off the power.

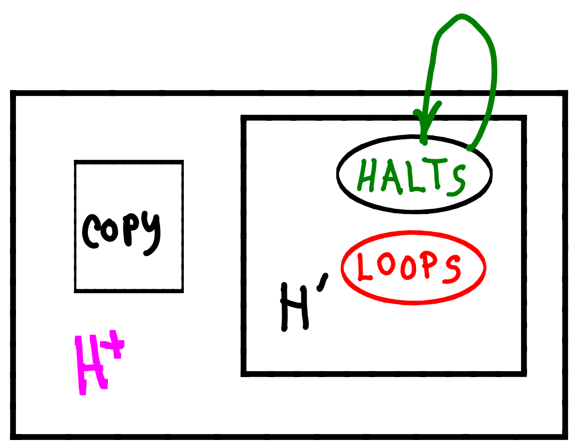
Consider putting **desc(H+) on H+**'s **input tape**. **What must happen?**



H+ first does exactly what **Copy** would do, **copy its input**.
Next, **H+ acts exactly as H would**.

The **tape is now** thought of as,
an **input**, **x = desc(H+)**, **followed by a machine description**, **desc(M)** = **desc(H+)**.

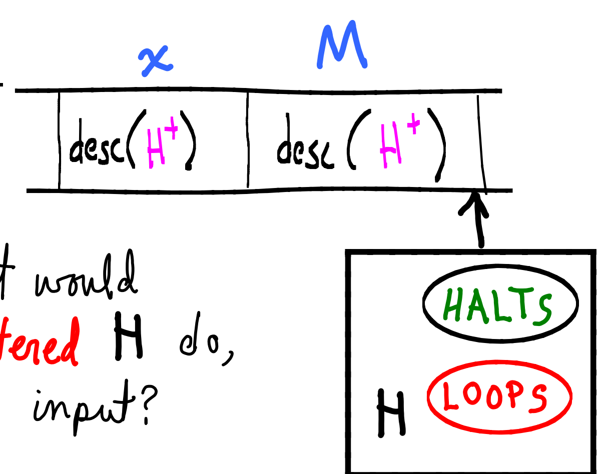
H+ WILL either (because **H** always halts in HALTS or LOOPS)
(**reach HALTS and then loop**)
OR
(**reach LOOPS and then halt**).



SUPPOSE H+ loops.

1. **H+ reached HALTS**.
2. Then **H** with input **xM** == **desc(H+) desc(H+)**, **would have halted** in **HALTS**.
3. BUT **H+ reading desc(H+) loops (our assumption)**.
4. Since **H is correct**, **it would not go to HALTS**.
5. **H+ cannot reach HALTS, and does not loop.**
6. This **contradicts our assumption** that **H+ loops**.

We **assumed H exists, i.e., it works correctly**.
Assuming also that H+ loops leads to a contradiction.
At least **one of these assumptions** must be **false**.

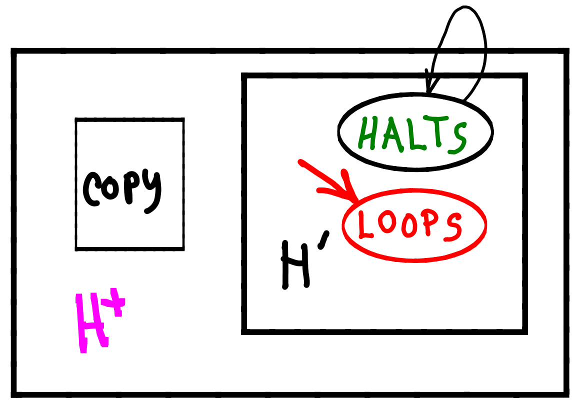


ask, what would the ***unaltered*** *H do, given this input?*

SUPPOSE H+ halts.

1. **H+ reached LOOPS.**
2. **H reading desc(H+) desc(H+) must reach LOOPS.**
3. BUT **desc(H+) H+ halts.**
4. **H is correct**; so, **H cannot reach LOOPS.**
5. **desc(H+) H+ cannot reach LOOPS.**

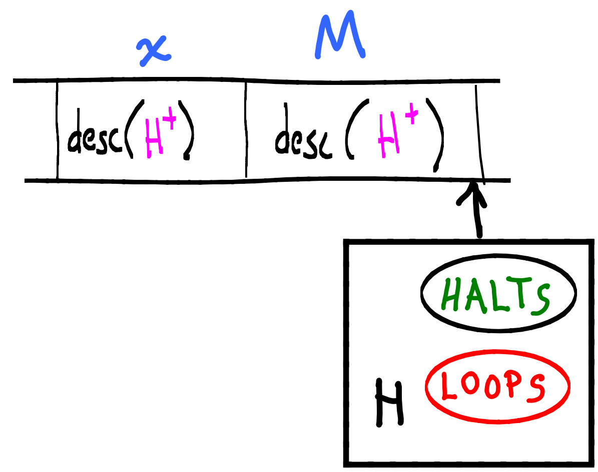
We assumed H is correct.
Assuming also that H+ halts leads to a contradiction.



If H exists, H+ exists, is a TM, and either halts or loops. (Building H+ from H was easy and resulted in a TM.)

But both cases (H+ either halts or loops) lead to contradictions.

The assumption that H exists must be false.



This is better than diagonalization: we have a real, uncomputable function. The function exists because every TM M either halts of loops forever, given an input x.

There is a function H() mapping
H: { xM } ===> {0, 1}
from positive integers to {0, 1}, but no TM can compute it.

Are we doomed?

Build something H- that partially computes the Halting Problem?

Works for some inputs, but not others?

Works for some fixed number of inputs?

Has a lookup table?

How many machines act exactly like any given description?

How many descriptions are there?

How many other things are not Turing computable? What does this say about cognition? ...???

Another Method? Does it work? Why?

```
Hnew( x, M)

   print "loops forever"

   1. Simulate xM for one step.
   2. If  xM halted
          print "halts"
      else
          go to 1.
```

$\Rightarrow$ Is HP computable?

Bottom Line

Suppose we try to write a program $H(x, M)$.

We **succeed** for some special cases $\{M_1, M_{25}, M_{300}, \ldots\}$

But, we always find a new M_i and have to rewrite $H(x, M)$. Also, we get it to work for $\{x_1, x_2, \ldots\}$, but find a new x_i for which $x_i M_j$ loops (or halts) (can we figure that out?), and our $H(x_i M_j)$ says it will loop. Back to re-writing our H().

HP $\Rightarrow$ We will never be bored!

Formal Proof

Notation: "[halts]" means "H+ halts when reading its own description"; "[loops]" is to be read similarly; "==>" means, "implies", in the logical sense of material implication; "-" means logical NOT.

1. (H exists) ==> (H+ exists (is a TM)) (by properties of TM)

2. (H+ exists) ==> [halts] OR [loops] (by properties of TM)

3. (H+ exists) ==> -[loops] AND -[halts] (demonstrated above)

4. (H exists) ==> ([halts] OR [loops]) AND (-[loops] AND -[halts]) (by 1. and 2.)

5. (H exists) ==> ([halts] AND -[halts]) OR ([loops] AND -[loops]) (by AND/OR properties)

6. p ==> q EQUALS -q ==> -p (by properties of "==>")

7. -(([halts] AND -[halts]) OR ([loops] AND -[loops])) ==> -(H exists) (by 5. and 6.)

8. -(([halts] AND -[halts]) OR ([loops] AND -[loops])) (true by AND/OR properties)

9. -(H exists) (syllogism applied to 7. and 8.)