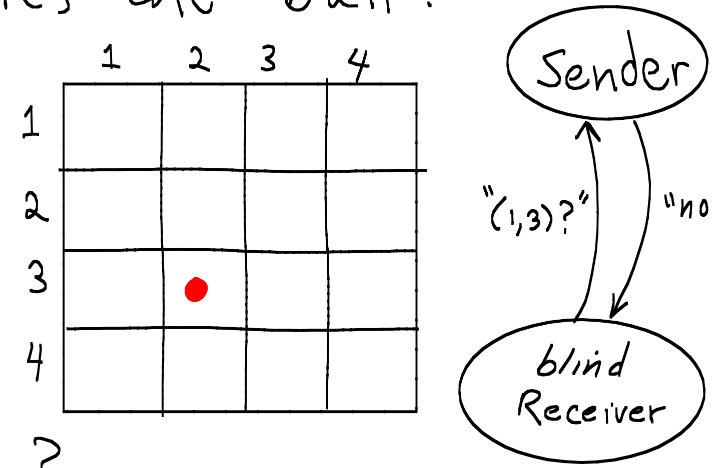
CODING and INFORMATION
We need encodings for data.

where's the ball?



- Ask yes/no questions.
- Are all questions-answers equally informative?
- Min number of questions?

How many questions must be asked to be certain where the ball is. (cases: avg, worst, best)

- How much do you learn from each yes/no?

- Is that a Good series of questions, "In (1,3)"?

Avg number of questions needed (assume equally likely boxes)?

P(Hit 1st) = 1/16

$\Rightarrow$ P(Hit 2nd) = P(Hit 2nd | Miss 1st)P(Miss 1st) = (1/15)(15/16) = 1/16

P(Hit 3rd) = (1/14) * P(Miss 2nd and 1st) = (1/14)(14/15)(15/16) = 1/16

E(n) = 1*(1/16) + 2*(1/16) + ... + 15*(1/16) + 15(1/16) = (1+2+3+...+15+15) / 16 = 8 1/2 - 1/16

- Different set of question?

"in (1,*) or (2,*)", "in (1,*)", "in (2,1) or (2,2)" ...

$\Rightarrow$ Each Q reduces space by ½ $\Rightarrow$ Exactly 4 questions

$\Rightarrow$ Most information if each Q-A splits possibilities 50-50.

Measure info content of answer?

prob(yes) = prob(no) $\Rightarrow$ $\log_2(\text{Prob}(\text{yes}))$

$= \log_2(2^{-1})$

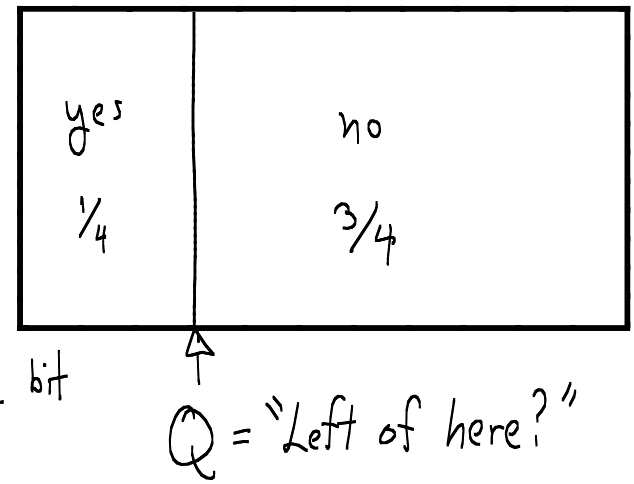
$= -1$ (hmm, make +?)

$\Rightarrow$ 1 bit

$\Downarrow$

$-\log_2(\text{Prob})$ = **info measure**

$\left.\begin{array}{l} \text{Prob(yes)} = 1/4 \\ \text{Prob(no)} = 3/4 \end{array}\right\}$ assume



yes: $\log(2^{-2}) = 2$ bits

no: $\log(3/4) \approx -\log(.7) \approx -\log(1/\sqrt{2}) = \frac{1}{2}$ bit

Avg info of answer?

$= \text{prob(yes)} \cdot (2 \text{ bits}) + \text{prob(no)} \cdot (\tfrac{1}{2} \text{ bit})$

$= \frac{1}{4}(2) + \frac{3}{4}\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$ bit

---

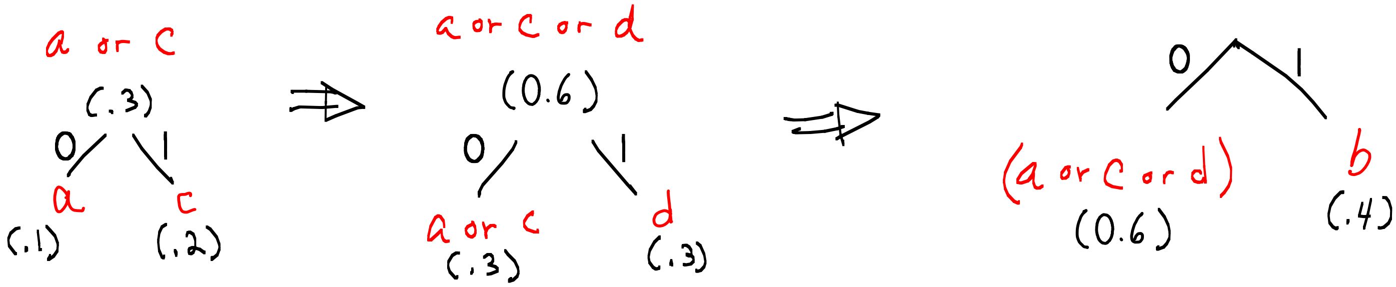
Extreme: $\text{Prob(yes)} = 1/2^{10}$ $\quad \text{Prob(no)} = (2^{10}-1)/2^{10} \approx 1$

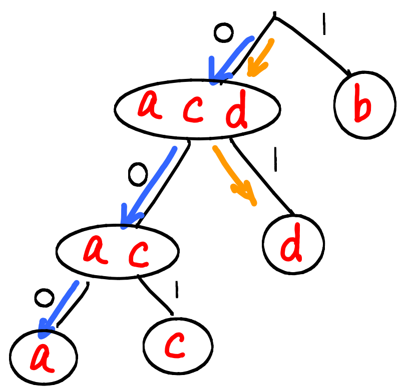
$\text{avg} = \frac{1}{2^{10}}(10 \text{ bits}) + (1)\underbrace{\log(1)}_{0 \text{ bits}} \Rightarrow 1/100$ bit

Thm max avg. if $p_i = \frac{1}{n}$ for n possibilities. $\left(\sum p_i = 1\right)$

How to encode to get 50-50? $\Rightarrow \begin{pmatrix}\text{Pair} \\ \text{least} \\ \text{Probable}\end{pmatrix} \sim$ equally likely.

(a, b, c, d) w/ prob. (.1, .4, .2, .3)





Huffman Code	msg
0 0 0	a
0 0 1	c
0 1	d
1	b

Shannon information = (Expected (Avg.) bits of information, per message) = $E(-\log \text{Prob})$

$$(\text{Entropy}, H) = -\left[\underset{a}{0.1\log(0.1)} + \underset{b}{0.4\log(0.4)} + \underset{c}{0.2\log(0.2)} + \underset{d}{0.3\log(0.3)}\right]$$

$$= [0.33 + 0.53 + 0.46 + 0.52] = 1.84$$

bits per message.

How'd we do?

$$\boxed{\text{Avg. \#bits sent, using our code.}} = \underset{a}{(0.1)3} + \underset{b}{(0.4)1} + \underset{c}{(0.2)3} + \underset{d}{(0.3)2}$$

$$= 0.3 + 0.4 + 0.6 + 0.6 = 1.9$$

bits per message.

We are sending more bits than information content, but we are very close.

MIN-Length code ==> MAX compression ==> most info bits in least number of communicated bits.

Suppose n different "messages" to send, n = 2^k.
Maximum entropy => equally likely: Prob(message-i) = (1/n) for any message-i.

Expected information per message is,

Sum[- (1/n) log[1/n]] = - n (1/n log[1/n]) = -1 log[2^-k] = -1 (-k) = k bits per message. If we use a k-bit code for our messages, we will be 100% compressed. (k-bit integers? Are they equally likely?)

Error Detection / Correction

"message" could be a bit, a string of bits, a character, a page of characters, ...

"message" → Communication Channel → "mfssage"

message coded in bits:

"1" → encode → 1 → channel → 0 → decode → "0"   1-bit Error not detectable

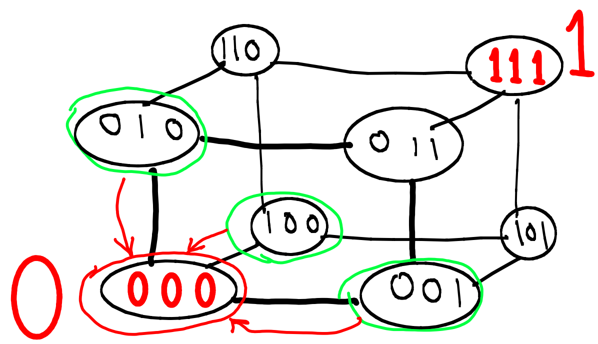
2-bit encoding

"1" → encode → 11 → channel → 01 → decode → Error detected

Code words: 00 and 11 --- "0" and "1"
Code words: 10 and 01 --- 1-bit errors: odd parity codeword indicates error.
Works for k-bit messages w/ 1 parity bit, but only if 2-bit errors very unlikely (never occur?).

1-bit Error Correction w/ 3-bit code words:
"0" ==> 000
"1" ==> 111

001 ==> "0"	011 ==> "1"
010 ==> "0"	101 ==> "1"
100 ==> "0"	110 ==> "1"



1-bit Correction, 2-bit Detection

-- odd parity: 1-bit error corrected

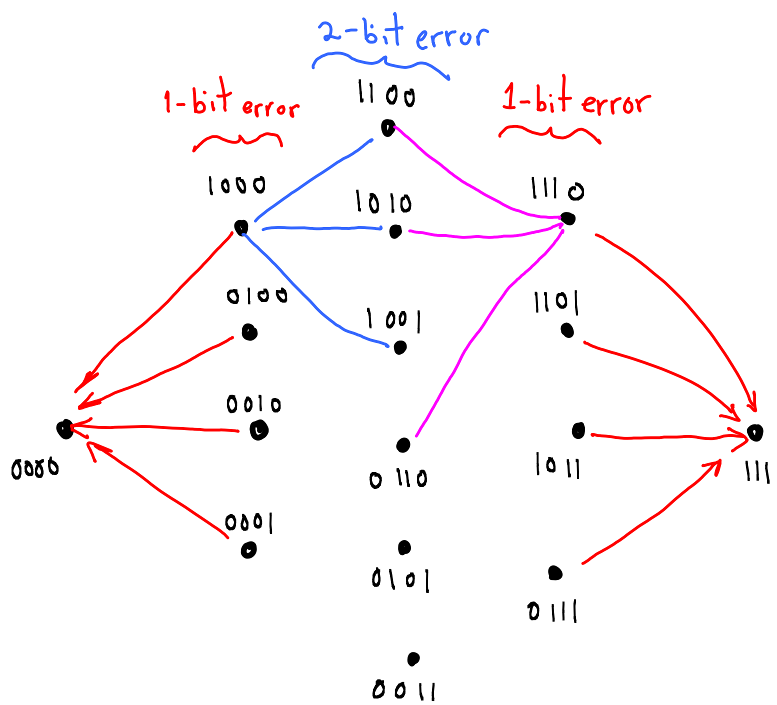
-- exactly two 1's: 2-bit error detected

-- otherwise: no error

How many extra bits are needed at minimum? Depends on noise in channel: Shannon Noisey Coding Theorem.

Can you think of a scheme like the parity-bit scheme that uses as few bits as possible? (See Reed-Solomon codes, for instance.)

More bits, higher error probability.



ALU, numbers

Positional notation
for numbers

d_i is a "digit", a symbol for a value: value("d_i")

b is a value, the "base" of the number notation.

There is a rule to find the value, given the symbols.

$$\text{value}(\text{"}d_n d_{n-1} \cdots d_1 d_0\text{"}) = d_n \cdot b^n + d_{n-1} \cdot b^{n-1} + \cdot \cdot \cdot \; d_1 \cdot b^1 + d_0 \cdot b^0$$

unsigned 3-bit binary
binary:
--- digits = { "0", "1" }
--- base = 2

$000 \rightarrow \text{value} \triangleq 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 0$

$001 \rightarrow \text{value} \triangleq 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 1$

$\cdots \qquad \cdots \qquad \cdots$

$111 \rightarrow \text{value} \triangleq 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 7$

oops, more symbols?

Let's do some arithmetic.
ADD:

$\begin{array}{r} 0 \\ +0 \\ \hline 0 \end{array} \quad \begin{array}{r} 0 \\ +1 \\ \hline 1 \end{array} \quad \begin{array}{r} 1 \\ +0 \\ \hline 0 \end{array} \quad \begin{array}{r} 1 \\ +1 \\ \hline 10 \end{array} \Rightarrow$

carry

A	B	S
0	0	0
0	1	1
1	0	1
1	1	0

XOR

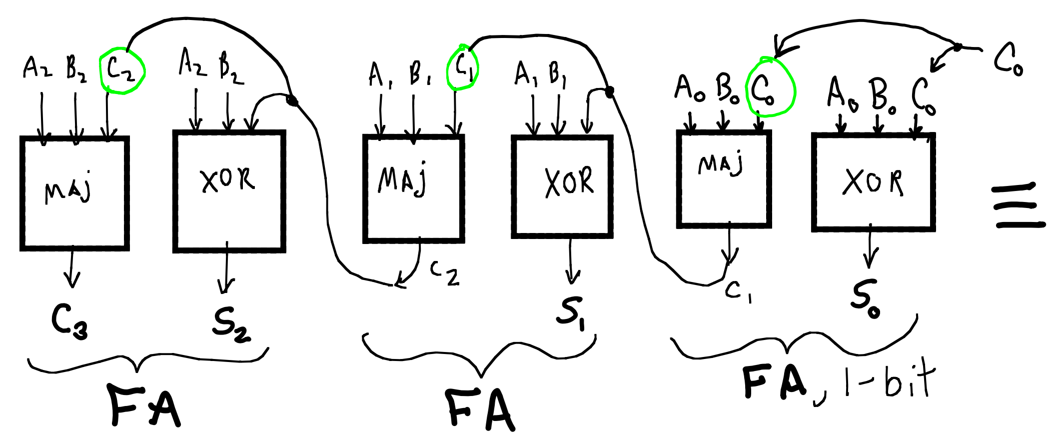
w/ **Carry In**:

$\begin{array}{r} 1 \\ 0 \\ +0 \\ \hline 1 \end{array} \quad \begin{array}{r} 1 \\ 0 \\ +1 \\ \hline 10 \end{array} \quad \begin{array}{r} 1 \\ 1 \\ +0 \\ \hline 10 \end{array} \quad \begin{array}{r} 1 \\ 1 \\ +1 \\ \hline 11 \end{array} \Rightarrow$

A	B	S
0	0	1
0	1	0
1	0	0
1	1	1

$\overline{\text{XOR}}$

$$A_2 A_1 A_0 + B_2 B_1 B_0 = C_3 S_2 S_1 S_0$$



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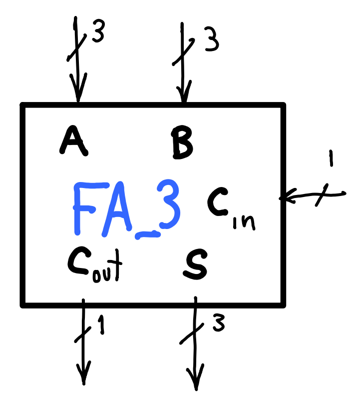


3-bit
FuLL Adder

Let's try
SUBTRACTION

$$A_2 A_1 A_0 - B_2 B_1 B_0 = b_3 \; S_2 S_1 S_0$$

possible borrow

But first, let's look at a single column of subtraction

All possible 1-bit subtractions

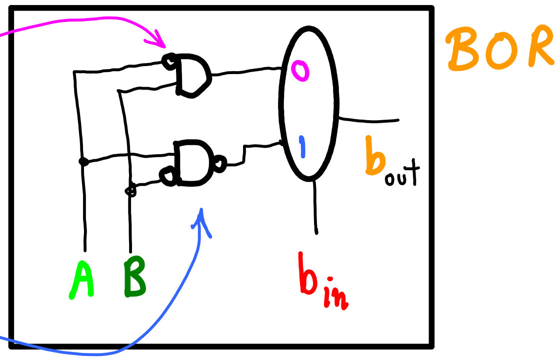
All possible single columns

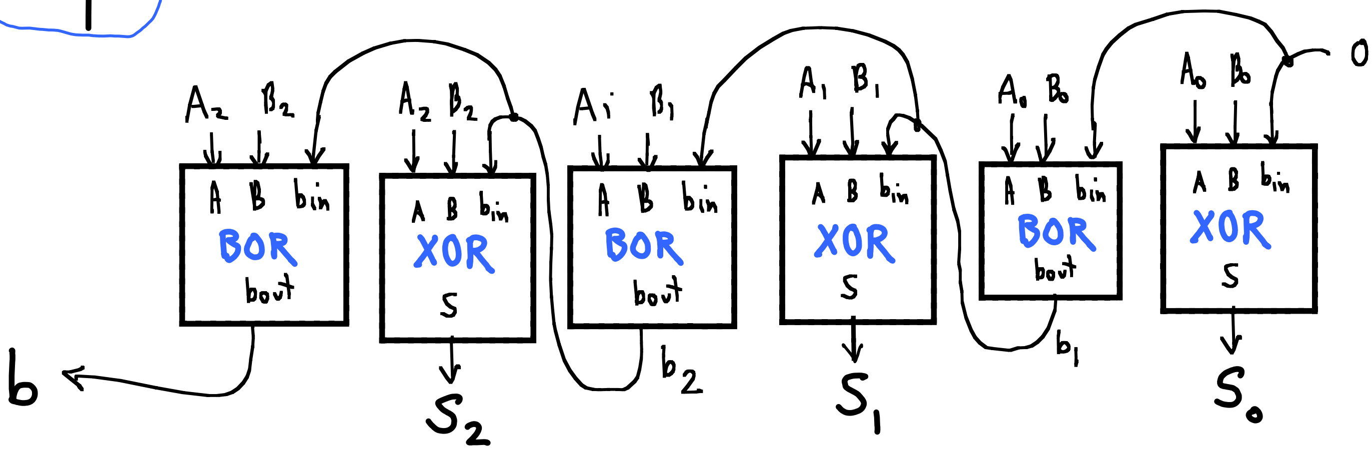
b_{in} (borrow in)	0	0	0	0	-1	-1	-1	-1	
$+A$	0	0	1	1	0	0	1	1	
$-B$	0	-1	0	-1	0	-1	0	-1	
S	0	1	1	0	1	0	0	1	$= XOR(A, B, b_{in})$
b_{out} (borrow out)	0	1	0	0	1	1	0	1	$= BOR(A, B, b_{in})$

b_{in}	A	B	b_{out}
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

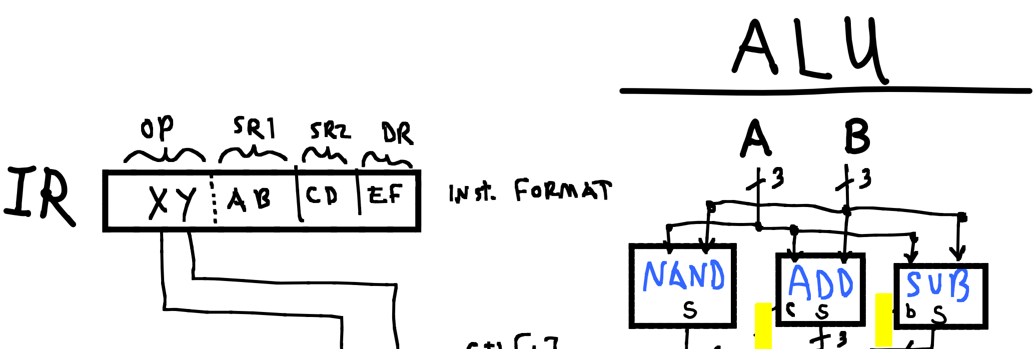
$(\bar{A}_i \bullet B_i)$

$(\overline{A_i \bullet \bar{B}_i})$





ALU

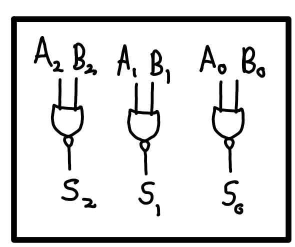


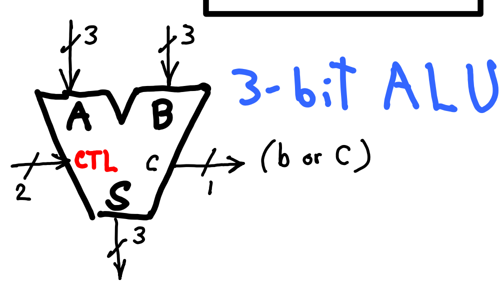
OP CODES

00	NAND
01	ADD
11	SUB

(simplified instruction, only shows ALU ops)

3-bit, bit-wise NAND

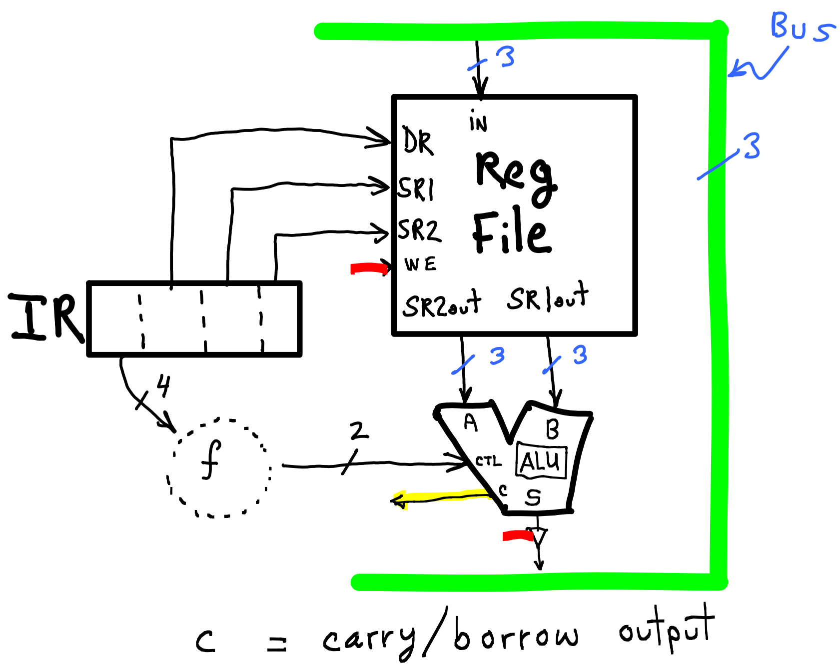




It's almost this simple in the LC3.

This is a **3-bit version** of LC3 (sort of).

Some sort of function *f* converts 4-bit opcode to 2-bit ALU.ctl. In uCoded control, this function is implemented as control bits in ROM.



Unsigned Arith. Errors

3-bit numbers

$A + B > 7 \implies c = 1$ $\quad S = (A+B) \bmod 2^3 \le 7$

$A - B < 0 \implies b = 1$ $\quad S = (A-B) \bmod 2^3 \ge 0$

$$\left(\underline{C_3 \cdot 2^3} + S_2 \cdot 2^2 + S_1 \cdot 2^1 + S_0 \cdot 2^0\right)_{\bmod 2^3} = S_2 S_1 S_0$$

$$\left(\underline{b_3(-1)\cdot 2^3} + S_2 \cdot 2^2 + S_1 \cdot 2^1 + S_0 \cdot 2^0\right)_{\bmod 2^3} = S_2 S_1 S_0$$

$b = c = 1 \implies$ Overflow Error

We have **8 possible 3-bit patterns** (**symbols**).

Choose an interpretation.

3-bit Code	interpretation as value
0 0 0	0
0 0 1	1
0 1 0	2
0 1 1	3
1 0 0	4
1 0 1	5
1 1 0	6
1 1 1	7

NB--We represent the value using another encoding: base 10!

What **other number values** are we interested in?

Are **other encodings** useful?

Scaled Numbers

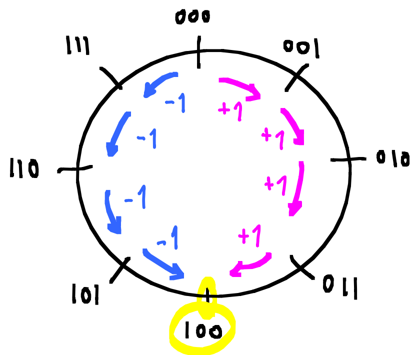
3-bit Code	interpretation as value
0 0 0	0
0 0 1	8
0 1 0	16
0 1 1	24
1 0 0	32
1 0 1	40
1 1 0	48
1 1 1	56

Sign-magnitude

3-bit Code	interpretation as value
0 0 0	+0
0 0 1	+1
0 1 0	+2
0 1 1	+3
1 0 0	−0
1 0 1	−1
1 1 0	−2
1 1 1	−3

2's-Complement Encoding:
represent **POSTIVE** and **NEGATIVE**

CODE	Value
0 1 1	+3
0 1 0	+2
0 0 1	+1
0 0 0	0
1 1 1	-1
1 1 0	-2
1 0 1	-3
1 0 0	-4



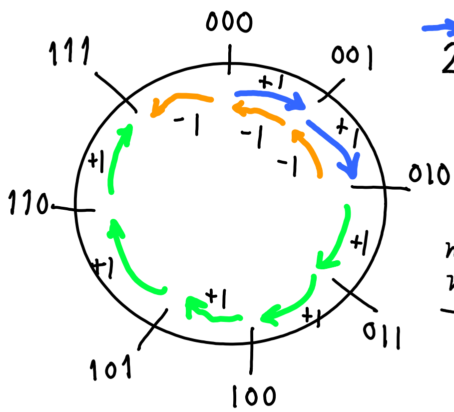
moving +4 ≡ moving −4
Which value makes sense?

sanity check: 0 - 1?

$$\begin{array}{rccc} & 0 & 0 & 0 \\ - & 0 & 0 & 1 \\ \hline (-1) & 1 & 1 & 1 \end{array}$$

-1 +10 -1 +10 -1 +10

hmm, kind of makes sense.



$2 - (+3)$

$2 + (-3)$

move +2
move +5

(-1)

$-k$ can be moving:

k steps

OR

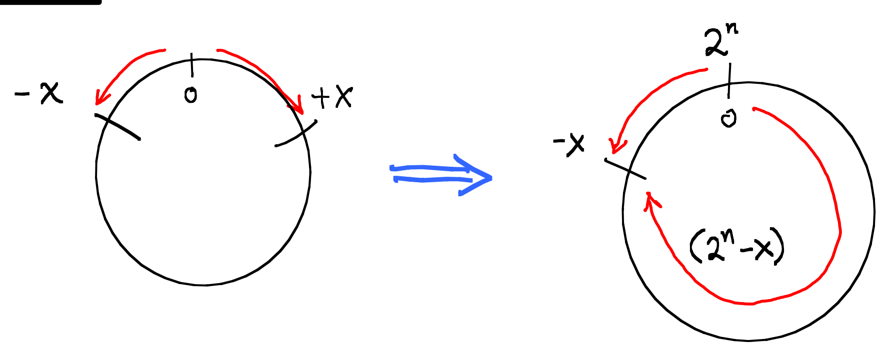
$2^n - k$ steps:

$-3 \Rightarrow$ 3 steps

$\Rightarrow 2^3 - 3 = 5$ steps

n-bit 2's Complement

$-x \rightarrow 2^n - x$



Sanity check $-(-x)$?

$-x \Rightarrow (2^n - x)$

$-(-x) \Rightarrow 2^n - (2^n - x)$

$= x$

$-(-X) = X$
in 2's comp.

Try $(-(-3))$ in $n=3$ 2's comp : $2^n = 2^3 = 8$

$(-3)_{2\text{'s comp}} \longrightarrow 2^n - 3 = 8 - 3 = 5$

$(-(-3)_{2\text{'s comp}})_{2\text{'s comp}} \longrightarrow 2^n - 5 = 8 - 5 = +3$

Converting To 2's Comp?

n-bit

How do we do this simply, in general?

$$
\begin{array}{rcl}
 & & \text{borrow} \quad {-1}\ 10 \quad {-1}\ 10 \quad {-1}\ \ldots \quad {-1}\ 10 \quad {-1}\ 10 \\
2^n & = & 1\ \ 0\ \ 0\ \ldots\ 0\ \ 0\ \ 0\ \ 0 \ldots 0 \\
-x & = & -\ x_{n-1}\ x_{n-2} \cdots x_{n-j}\ \boxed{1}\ \ 0\ \ 0 \ldots 0 \\
\hline
S & = & S_{n-1}\ S_{n-2} \cdots S_{n-j}\ \boxed{1}\ \ 0\ \ 0 \cdots 0
\end{array}
$$

Notice: The 1st **non-zero** bit of x gets copied to S:
borrow = 10
subtract bit = - 1
----
sum bit = 1

Note: columns with borrows give a bit flip.

$$
\begin{array}{r}
1 \\
-\,x_k \\
\hline
\overline{x}_k
\end{array}
$$

$$
\begin{array}{rcl}
2^n & = & 0\ \ 1\ \ 1\ \ldots\ 1\ \ \overset{10}{0}\ \ 0\ \ 0 \ldots 0 \\
-x & = & -\ x_{n-1}\ x_{n-2} \cdots x_{n-j}\ 1\ \ 0\ \ 0 \ldots 0 \\
\hline
(2^n - x) & = & \overline{x}_{n-1}\ \overline{x}_{n-2} \cdots \overline{x}_{n-j}\ 1\ \ 0\ \ 0 \cdots 0 \\
(2^n - x) - 1 & = & \underbrace{\overline{x}_{n-1}\ \overline{x}_{n-2} \cdots \overline{x}_{n-j}\ 0\ \ 1\ \ 1 \cdots 1} \\
 & & \\
x & = & x_{n-1}\ x_{n-2} \cdots x_{n-j} \cdots x_2\ x_1\ x_0
\end{array}
$$

-1

+1

negate

NB--These are the **negated bits** of x.

To Get 2sComp(x):

Negate bits,
add 1.

Produce - x in 2's Complement (regardless of whether x is + or -):

Negate bits (aka 1's Complement), then **add 1**.

Simple logic: inverter on each bit, carry in to lowest FA set to 1.

==> We can use **adder for signed subtraction**

Let's try

2's Comp of neg. number (expressed in 2's comp).

$2\text{'s Comp}(\underbrace{1\,x_3\,x_2\,x_1\,x_0})$ — a neg. number in 2's comp.

$0\,\overline{x}_3\,\overline{x}_2\,\overline{x}_1\,\overline{x}_0 + 1$

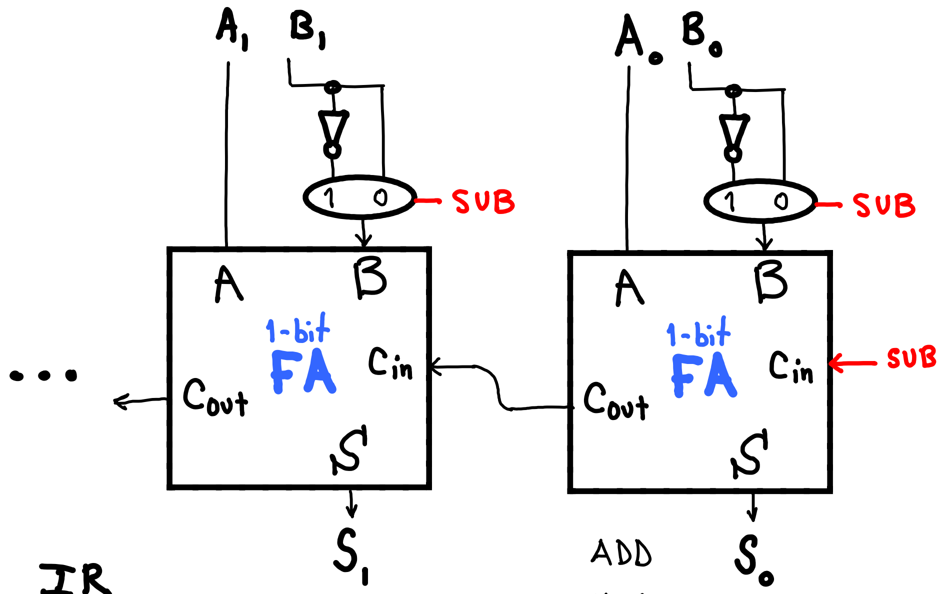
flip bits, add 1

	Least neg.	between	most neg.	
	1111	1abc	1000	
flip ⇒	0000	$0\,\bar{a}\,\bar{b}\,\bar{c}$	0111	⇐ flip
+1 ⇒	0001	0xyz	1000	⇐ +1
			?	

Extreme case ✓ok

in-between case ✓ok

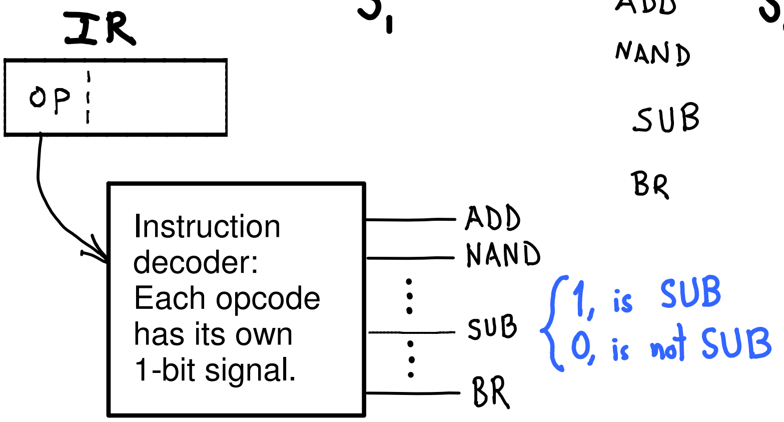
Extreme case ※ oops! what's wrong?



A + 2sComp(B)

sub = { 0, ADD ; 1, SUB }

ADD
NAND
SUB
BR



e.g., 3-bit (3 − 1)

A = 3 ⟹ 011

B = 1 ⟹ 001

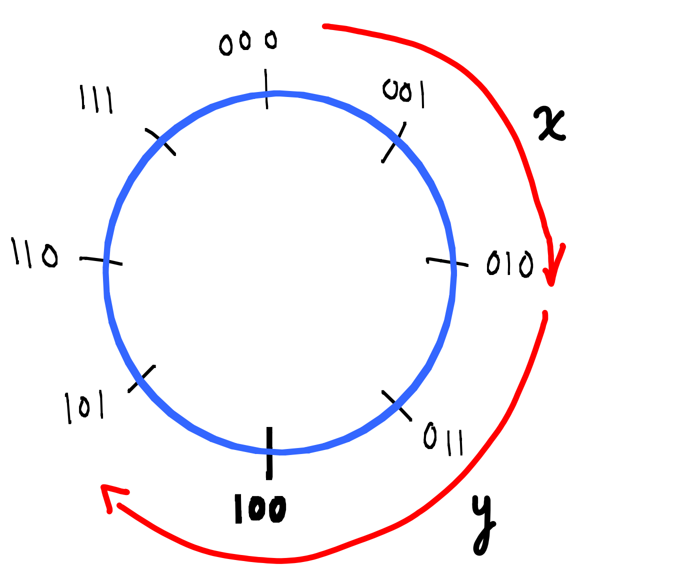
2sComp(B) ⟹ 110 + 1 = 111

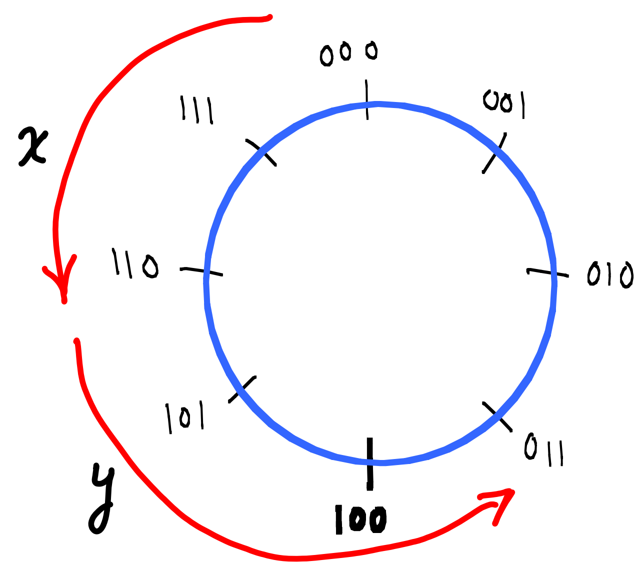
carries: 1 1 1 1

```
   1 0 1 1  (A)
 +   1 1 1  (-B)
 ----------
 (1) 0 1 0
```

Pos 2

2's complement Arith., Overflow





x > 0, y > 0
and x+y < 0

```
0xxx
0yyy
---------
1sss
```

x > 0, y < 0
and x-y < 0

```
  0xxx
- 1yyy  => 0zzz*
----------
  1sss
```

2's comp

x < 0, y <0
and x+y > 0

```
1xxx
1yyy
----------
0sss
```

x < 0, y > 0
and x-y > 0

```
  1xxx
- 0yyy  => 1zzz
----------
  0sss
```

2's comp

* could be 1000

ERROR = Same signs in, diff. out

Multiply

does $2x$ = Left shift?

$$\begin{array}{r} 1 \\ \times 10 \\ \hline 10 \end{array} \qquad \begin{array}{r} 10 \\ \times 10 \\ \hline 100 \end{array} \qquad \begin{array}{r} 100 \\ \times 10 \\ \hline 1000 \end{array}$$

1ST 1

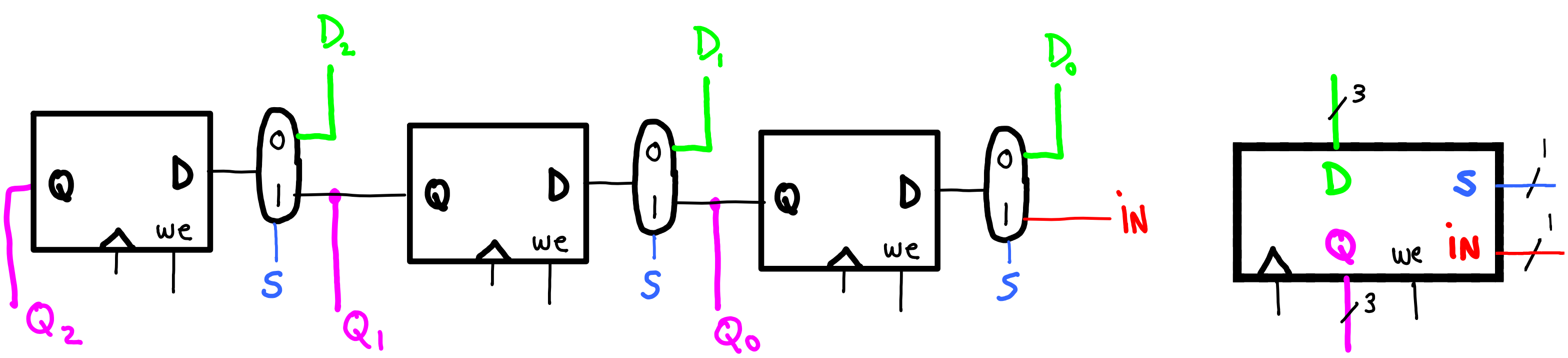
$$2x = x + x \Longrightarrow \begin{array}{rccccccccc} & x_n & x_{n-1} & \cdots & x_k & 1 & 0 & 0 & \cdots & 0 \\ + & x_n & x_{n-1} & \cdots & x_k & 1 & 0 & 0 & \cdots & 0 \\ \hline C_{n+1} & S_n & S_{n-1} & \cdots & S_k & 0 & 0 & 0 & \cdots & 0 \end{array}$$

Col. j	$x_j = 0$	$x_j = 1$
$\begin{array}{rr} & c_j \\ & x_j \\ + & x_j \\ \hline c_{j+1} & s_j \end{array}$	$\begin{array}{rr} & c_j \\ & 0 \\ + & 0 \\ \hline c_{j+1} & s_j \end{array}$	$\begin{array}{rr} & c_j \\ & 1 \\ + & 1 \\ \hline c_{j+1} & s_j \end{array}$

$$\Longrightarrow \left\{ \begin{array}{l} s_j = c_j \\ c_{j+1} = x_j \end{array} \right\} \Longrightarrow \underbrace{s_{j+1} = c_{j+1} = x_j}_{\text{left shift}}$$

x == 0 :   S == carry in; carry out == 0.
x == 1 :   S == carry in; carry out == 1.

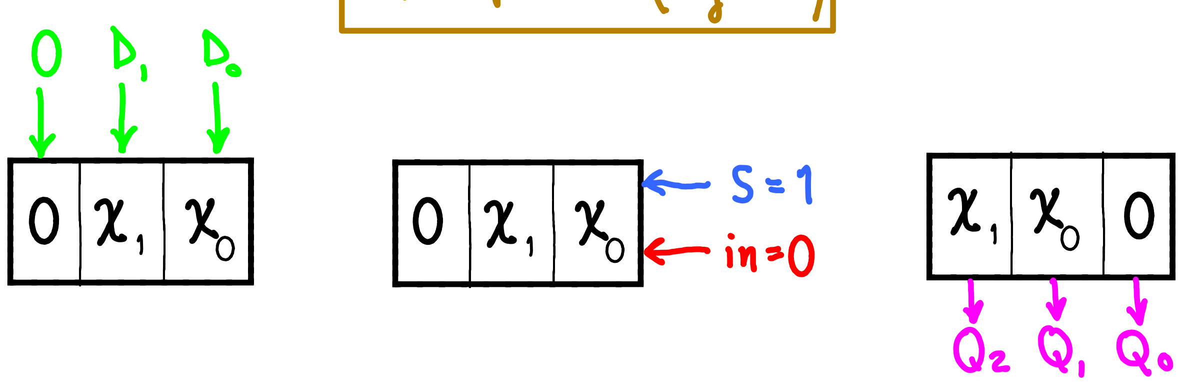
3-bit, parallel load, left-shift



Parallel write/load : we=1 and **S=0**:   Q[2:0] <==   D[2:0]   after next clock tick.

Shift Left :   **we=1** and **S=1**:   Q[2:0] <== { Q[1:0], **IN** }   after next tick.

Multiplier (by 2)



What about signed numbers?

Convert to unsigned.

Multiply.

Convert back.

$$x \times 7 == x_n\, x_{n-1} \ldots x_1\, x_0 \times (4+2+1)$$

$$==$$

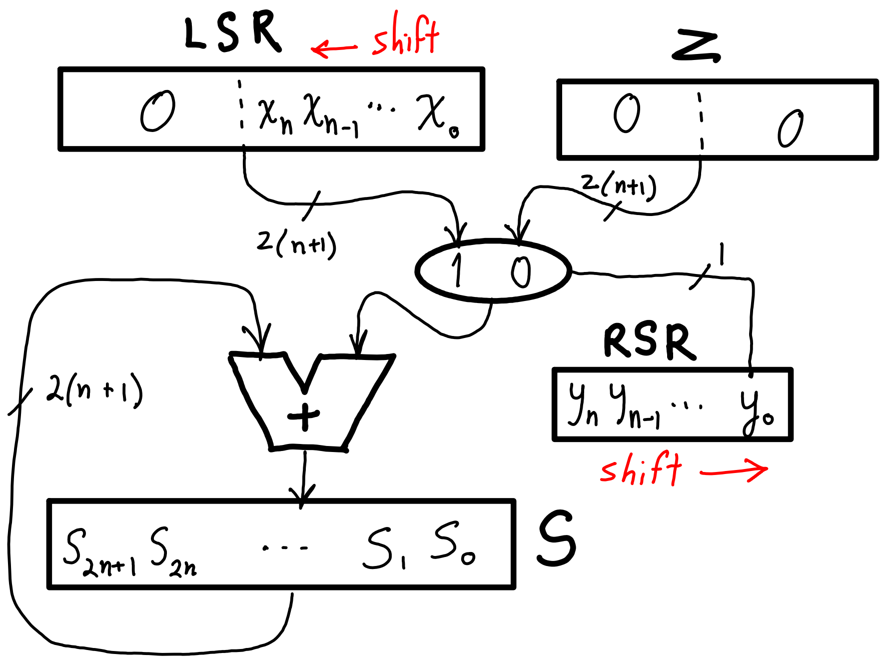
$$
\begin{array}{l}
\quad x_n\, x_{n-1} \ldots x_1\, x_0 \times (1) \leftarrow \text{0 left-shift} \\
+\, x_n\, x_{n-1} \ldots x_1\, x_0 \times (2) \leftarrow \text{1 left-shift} \\
+\, x_n\, x_{n-1} \ldots x_1\, x_0 \times (4) \leftarrow \text{2 left-shifts}
\end{array}
$$

$$==$$

$$
\begin{array}{rl}
 & x_n\, x_{n-1} \ldots x_1\, x_0 \\
+ & x_n\, x_{n-1} \ldots x_1\, x_0\, 0 \\
+ & x_n\, x_{n-1} \ldots x_1\, x_0\, 0\, 0
\end{array}
$$

MULTIPLY:
LSR: partial products, initially x.
S: partial sum, initially 0.
RSR: initially y.
Z: all 0s

RSR's low-bit MUXes Z or LSR to adder.



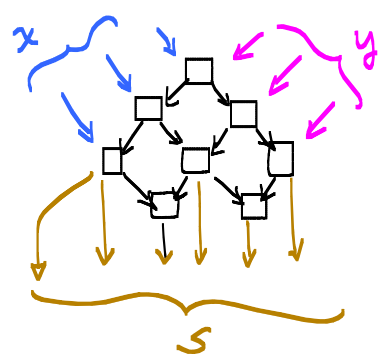
What if y has a 0 bit? Then add 0 instead of shifted x: e.g., y = 0...101 add 0, not **B**.

$\boxed{\text{e.g. } 1011 \times 0101}$

$$
\begin{array}{rcccccccc l}
 & & & & & 1 & 0 & 1 & 1 & (x = 11) \\
 & & & & \times & 0 & 1 & 0 & 1 & (y = 5) \\
\hline
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \leftarrow \text{START} \\
+ & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & \leftarrow \text{0-shift, } y_0 = 1 \\
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \leftarrow \text{1-shift, } y_1 = 0 \\
+ & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \leftarrow \text{2-shift, } y_2 = 1 \\
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \leftarrow \text{3-shift, } y_3 = 0 \\
\hline
 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & (S = 32 + 16 + 7 = 55)
\end{array}
$$

Can we simplify multiplier?
-- Get rid of zero register and mux.
-- Use y_i to write-enable S register.

Can we speed up multiply? We currently iterate n times to multiply n-bit numbers. Add more hardware? How?

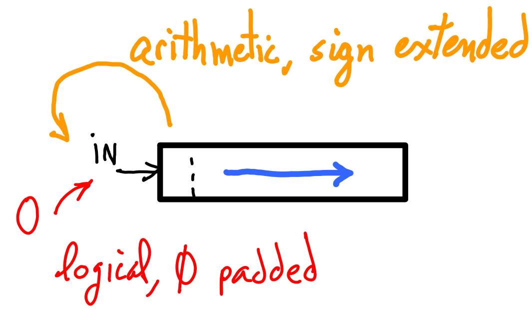


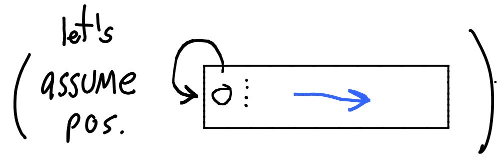
left-shift(y) == y X 2 ===> **right-shift(y) == y / 2**

Ok, for division by a power of 2.

$\boxed{\text{Div by 2}}$

R-Shift:





Integer Division = drop remainder

011 R-shift → 001

3 ÷ 2 = 1

0111 $(\text{R-shift})^2$ → 0001

7 ÷ 4 = 1

R-shift(n) = divide-by-2^n

If divisor is not power of 2?

$x = k \cdot q + r$ { k is divisor, q is quotient

$q = \#\, k\text{s in } x$

x: | k | k | k | k | k | r |

divBySubtraction(x, k)

```
q = 0;
while( x >= k )
    q++
    x = x - k
endWhile
```

time $= O(q)$

divByAddition(x, k)

```
q = 0;   y = 0
while ( x - y >= k )
    q++
    y = y + k
endWhile
```

We'd like $O(\log q)$ = # bits of q ⟹ long division

1. **Try** *n-th* power of 10, $d_n00\ldots0$

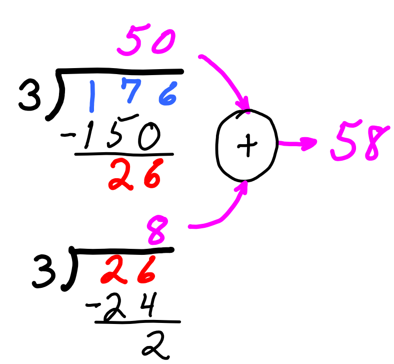
   subtract: $x - k \times d_n00\ldots0$

2. **result** x **non-negative?**

   yes: **save** $d_n00\ldots0$, x <=== x

3. **next power of 10:** n <=== n-1

$$\begin{array}{r|l} & d_n00\ldots0 \\ \hline k & x \\ & -k \cdot d_n00\ldots0 \\ \hline & x \end{array}$$



INTEGER (unsigned) DIVISON

$x = kq + r$ **k = divisor**, **q = quotient**, r = remainder (ignore for now). **FIND q.**

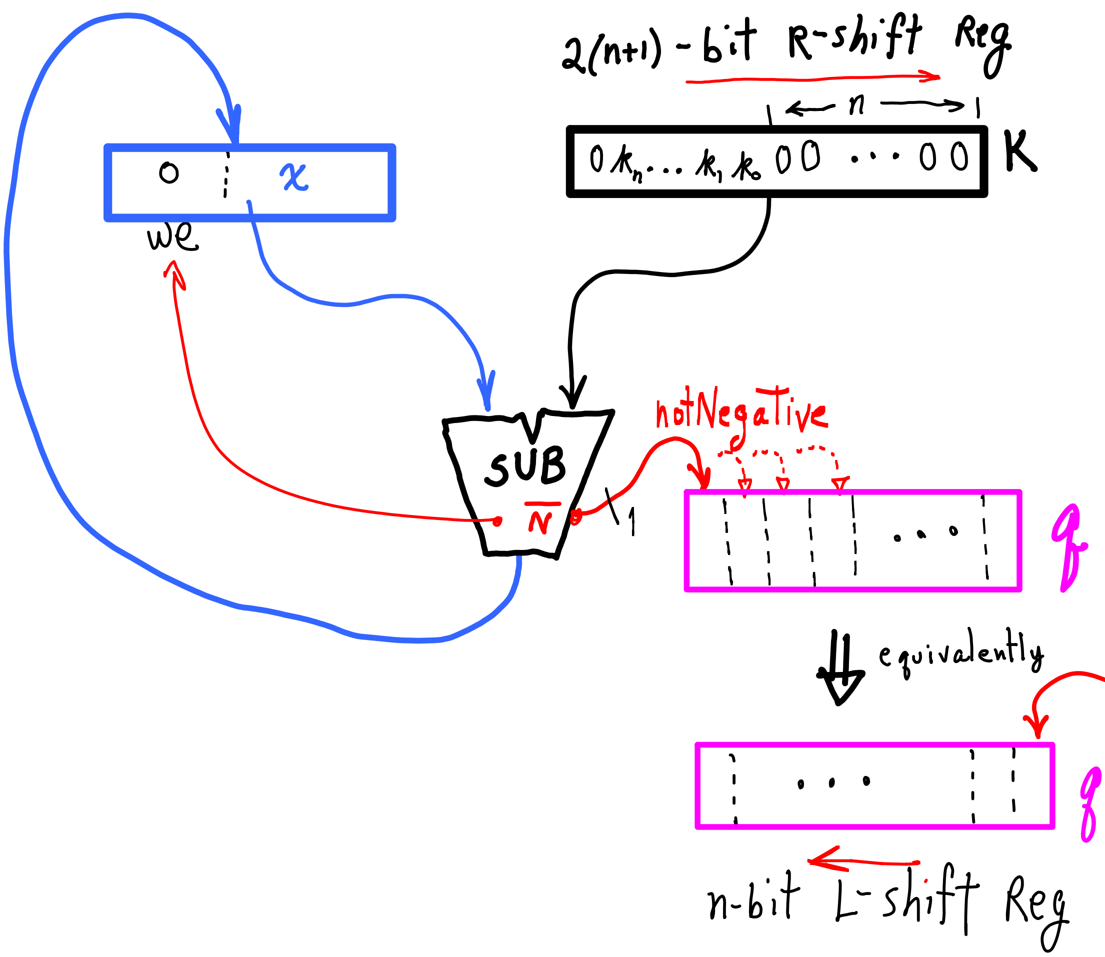
$$x = k\,q = k\,q_n 2^n + k\,q_{n-1} 2^{n-1} \cdots + k\,q_0 2^0$$

$$\left(x - \underbrace{k \cdot \overset{\text{try } q_n = 1}{1} \cdot 2^n}_{k \text{ L-shifted } n}\right) \geq 0 \ ? \quad \begin{cases} \text{yes: } q_n = 1 \\ \text{no: } q_n = 0 \end{cases} \rightarrow \boxed{q_n \mid 0 \mid 0 \mid 0 \mid 0 \mid \quad \mid 0 \mid 0} \ q$$

0 or 1

$$x \leftarrow k\,q_{n-1} 2^{n-1} + \cdots + k\,q_0 2^0 \Leftarrow x \leftarrow x - (k\,q_n 2^n)$$

$$\left(x - \underbrace{k \cdot \overset{\text{Try } q_{n-1} = 1}{1} \cdot 2^{n-1}}_{k \text{ L-shifted } (n-1)}\right) \geq 0 \ ? \quad \begin{cases} \text{yes: } 1 \\ \text{no: } 0 \end{cases} \rightarrow \boxed{q_n \mid q_{n-1} \mid 0 \mid 0 \mid \cdots \mid 0 \mid 0 \mid 0} \ q$$

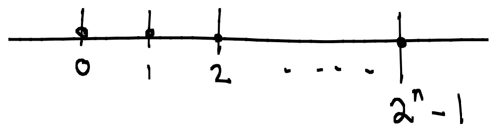


Move ***notNegative*** **RIGHT** one bit position after each SUB.

Move ***q*** **LEFT** one bit position after each SUB.

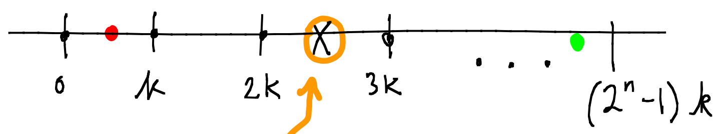
(write is to lowest bit position)

Floating Point



n-bit integers, range $= 2^n$
no discretization error

K-scaled integers: n-bit integer **x** represents k***x**, range = k*2^n.



this can't be represented, error $\approx k/2$

discretization error $\approx k/2$

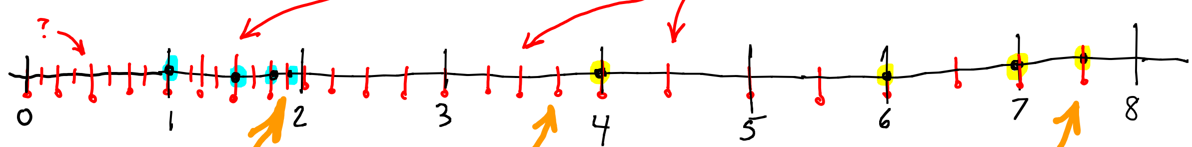
Near • k, % error $\Rightarrow \frac{k/2}{k} = 50\%$

Near • $k(2^n-1)$ $\Rightarrow \frac{k/2}{k(2^n-1)} \cong 1/2^{n+1}$

FP, exponential scaling

$2^m(1.XYZ)$

Can we get more consistent errors?



$2^0(1.111)$

$2^1(1.111) = (2+1+\frac{1}{2}+\frac{1}{4})$

$2^2(1.111)$

$2^0(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8})$

$\Rightarrow \underbrace{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}_{\approx 2} \pm(\frac{1}{16})$

$\Rightarrow$ error $\approx \frac{(1/16)}{2} = 1/32$

We can't represent every number.
We choose what type of errors to live with.

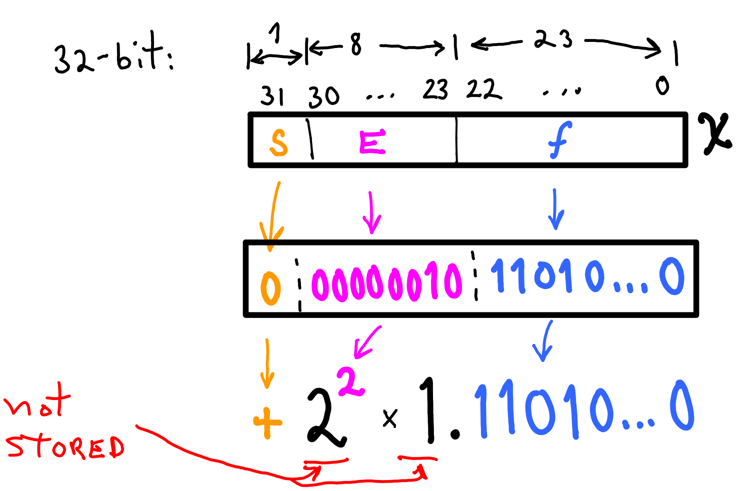
The part inside "(...)" is essentially integer.
The exponent determines the scaling.

===> geometrical-progression scaled integers

$\underbrace{4+2+1+\frac{1}{2}}_{\approx 8} \pm(\frac{1}{4})$

error $\approx \frac{(1/4)}{8} = \frac{1}{32}$

FP Format, single Float



value$(x) = S\, 2^E \times 1.f$

S = 0 : +
S = 1 : −

E pos.?
E neg.?
Use 2's comp for E ?

$\Rightarrow +2^2 \times (1+\frac{1}{2}+\frac{1}{4}+\frac{1}{16})$

Represent 0?

$$\boxed{0 \,|\, 0 \cdots 0 \,|\, 0 \;\; \ldots \;\; 0}$$

$$+2^{0} \times 1.0 \cdots 0 \Rightarrow 1\,?$$

If we used 2s-Comp for E, what's the smallest: 8-bit, 1000 0000 = −128

$$2^{-128} \times 1.00\ldots0$$ that's small. Do we need it?

$$\boxed{s \,|\, 10000000 \,|\, 00 \cdots 00}$$ means 0

Well, maybe we can live with that?
We have to stop somewhere.

How many bits do we need for 4 decimal digits of precision?

2 digits

6.023×10^{23} a nice number

4 digits

3 bits per digit $(2^3 \to 0..7)$
4 bits per digit $(2^4 \to 0..15)$

≈ 3.5 bits per digit

$$4 \text{ dig.} \times \frac{3 \text{ bit}}{\text{digit}} = 12 \text{ bits}$$

$$4 \text{ dig.} \times \frac{4 \text{ bit}}{\text{digit}} = 16 \text{ bits}$$

$$\Rightarrow 12 \le \begin{pmatrix}\text{Precision} \\ \text{of} \\ f\end{pmatrix} \le 16$$

(23 bits are enough)

---

range of E?
(2 dec. digits)

$$2 \text{ dig.} \times \frac{3 \text{ bit}}{\text{dig}} = 6 \text{ bits}$$

$$2 \text{ dig.} \times 4 \frac{\text{bit}}{\text{dig.}} = 8 \text{ bits}$$

$$6 \le \begin{pmatrix}\text{bits} \\ \text{of} \\ E\end{pmatrix} \le 8$$

Let's check

$$10^{23} \approx 8^{23} = (2^3)^{23} = 2^{69}$$

E
how many bits do we need for E?

E = 69, how many bits needed?

$$\text{Log}(69) \approx 1 + \text{Log}(64) = 7$$

Sorting is most common operation for numerical data

Checking x > y seems hard for floats.

Checking n > m for ints: do (n - m) and check sign bit, if 0 then True.

Can we check x > y using integer hardware?

That is, can we treat x and y as if they were integers, and do integer subtraction?

sign bit looks ok,

x	0	
y	1	

⇒ Looks like + 2's-comp

⇒ Looks like − 2's comp

⇒ ALL (+) floats are greater than all (−) floats, as 2's-comp.

How about the exponent part? $x - y$ as 2's_comp

2's comp exp

x	0	A	0 0
y	0	B	0 0

A = 011
B = 111

3-bit, 2s-comp exp

+exp	011
	010
	001
	000
−exp	111
	110
	101
	100

these are larger as unsigned, darn.

$$x = +2^{3} \cdot (1.00 \cdots 0)$$
$$y = +2^{-1} \cdot (1.0 \dots 0)$$

⇒ $(0\ 011\ 00 \dots 0)$ x
$(0\ 111\ 00 \dots 0)$ y

y looks like bigger 2's comp. number than x.

OOPS!

→ Let's see if we can patch this up.
Recall, our only problem is if both x and y have the same sign.

suppose sign(x) = sign(y)

(mag. comarison)

x | 0 | E_2 | f_2 |

y | 0 | E_1 | f_1 |

or

x | 1 | E_1 | f_1 |

y | 1 | E_2 | f_2 |

(Reverse result for neg.)

(for signs ≠ result is obvious)

let e_i = value(E_i)

Note: $e_1 > e_2 \Rightarrow 2^{e_1}(1.f_1) > 2^{e_2}(1.f_2)$

regardless of the fractional parts.

Let's check

Suppose $e_2 = e_1 + 1$

$1.111 \ldots 1$

$\underbrace{2^{e_2} \cdot (1.0)}_{\text{Smallest possible } x}$ $\underbrace{2^{e_1}(2 - \varepsilon)}_{\text{Largest possible } y} = 2^{e_1+1}(1 - \varepsilon/2)$

$= \underline{2^{e_2}(1 - \varepsilon/2)}$

Y is less than X

So, make E_1 look bigger than E_2

---

what we have so far

8-bit exponent (single float)

2's complement →

0111 1111 ($2^7 - 1 = 127$)

⋮

0000 0000 (0)

1111 1111

1111 1110

⋮

1000 0001 (-127, Not used, reserved for signal)

1000 0000 (-128, Not used, signals 0 ?)

Let's fix Exp

E.G., **3-bit exponents in 2s-complement**

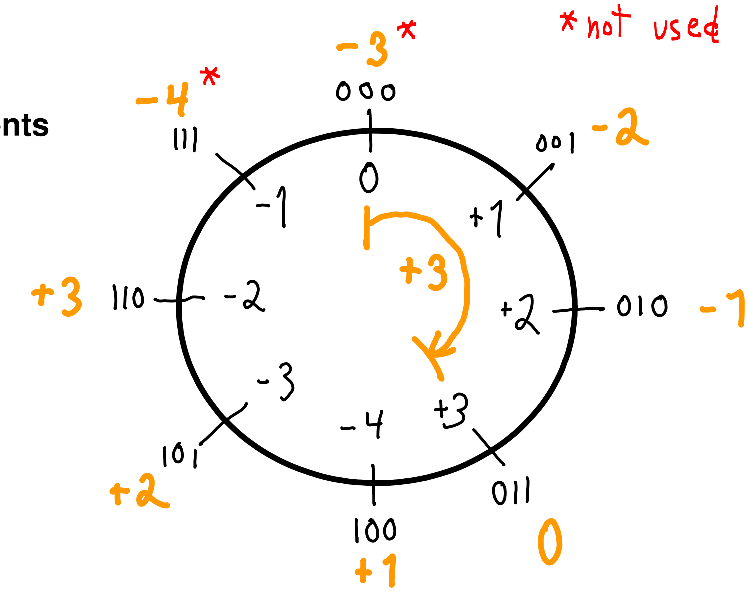
Negative **exponents** look smaller than positive **exponents** AS **unsigned ints**.

s	E	f

E = **e** + 011

value	**e** in 2s-comp			**E** in excess-3	
+3	011	+011	==>	110	
+2	010	+011	==>	101	
+1	001	+011	==>	100	
0	000	+011	==>	011	
-1	111	+011	==>	010	
-2	110	+011	==>	001	*
-3	101	+011	==>	000	*
-4	100	+011	==>	111	*

* These codes are reserved for special uses.
The exponent values -3 and -4 are not allowed.



Rotate 2'comp so that +3 becomes largest number available.

How to represent 0?

e	
111	0

→ ±∞

e	
000	0

→ 0 (±0)

e	
111	$f \neq 0$

→ NaN

e	
000	$f \neq 0$

→ not normalized

$2^{-3}\ 0.f$

8-bit FP, ADD

	s	E	f
x	0	001	0110
y	0	110	0010

001 − 011 = 110 → −2

Convert from excess-3 to 2's comp.

110 − 011 = 011 = +3

0	110	0110
0	011	0010

$+2^{-2} \times 1.0110$

$+2^{3} \times 1.0010$

Shift/align exponents

$2^3 \times 0.000001011\,0$

$2^3 \times 1.0010$

TRUNCATION?

add f part

$2^3 \times 1.001010$

rounding

round to nearest

$2^3 \times 1.0010$ → normalize → $2^3 \times 1.0010$ → encode

no shift needed in this example

0 | 011 | 0010 → To excess-3 → 0 $\underbrace{0 \mid 110 \mid 0010}_{y}$

$X + y = y$!?! ⇒ discretization, truncation, rounding

→ ERRORS be careful!

8-bit FP, MULT

0 001 1010 → 0 110 1010 → $+2^{-2} \times 1.1010$

0 110 0110 → 0 011 0110 → $+2^{3} \times 1.0110$

Convert E_s from excess-3 To 2s Comp

add exponents

check for overflow

2^1

Shift and MULT as unsigned INTs. Keep exp. To normalize

$$
\begin{array}{rl}
11010 & (\times 2^{-4}) \\
\times\ 10110 & (\times 2^{-4}) \\
\hline
+110100 & \\
+11010 & \\
+110100 & \\
\hline
1000111100 & (\times 2^{-8})
\end{array}
$$

normalize

$2^1 \times 1.000111100$

add exponents

$2^2 \times 1.00011110$

round To nearest

$2^2 \times 1.0010$

encode

0 010 0010

excess 3 code

0 101 0010

Convert to 32-bit FP   28

1. Convert to binary

$$
\begin{array}{rlcr}
28 & & & \\
-16 & \rightarrow 1\cdot 2^4 & = & 10000 \\
\hline
12 & \quad + & & \\
-8 & \rightarrow 1\cdot 2^3 & = & 1000 \\
\hline
4 & \quad + & & \\
 & \rightarrow 1\cdot 2^2 & = & 100 \\
\hline
 & & & 11100
\end{array}
$$

2. Normalize

$11100. \Rightarrow 2^4 \times 1.1100$

$e = 4$

3. Encode

	8-bits	23-bits
0	0000 0100	1100...0
+	$e=4$	1.1100

4. Convert e to excess $(2^{n-1}-1)$

excess $(2^{8-1}-1 = 127)$

$$
\begin{array}{rl}
00000100 & = e \\
+\ 01111111 & = 127 \\
\hline
10000011 & = E
\end{array}
$$

5. Replace e with E

0	10000011	1100 ...	0

Convert back

1. decode: $+\ 2^{10000011} \times 1.1100\ldots 0$

2. convert E

$$
\begin{array}{rl}
 & \leftarrow \text{borrows} \\
 & 10000011 \\
- & 01111111 \\
\hline
 & 00000100 = 4 = e
\end{array}
$$

$\Rightarrow 2^4 \times 1.11$ (convert f / mult. by 2^e) $\Rightarrow 1.1100 = 11100$

$\times 2^4$

(convert to dec.) $\Rightarrow 16 + 8 + 4 = 28$

- Problem with both truncation and rounding
  - They cause errors to **accumulate**
  - E.g., if always round up, result will gradually "crawl" upwards
- One solution: **round to nearest even**
  - If un-rounded LSB is 1 → round up (01**1** → 10)
  - If un-rounded LSB is 0 → round down (00**1** → 00)
  - Round up half the time, down other half → overall error is stable
- Another solution: **multiple intermediate precision bits**
  - IEEE 754 defines 3: guard + round + sticky
    - Guard and round are shifted by de-normalization as usual
    - Sticky is 1 if any shifted out bits are 1
  - Round up if 101 or higher, round down if 011 or lower
  - Round to nearest even if 100

- Suppose you added two numbers and came up with
  - 0 101 **11111 101**
  - What happens when you round?
  - Number becomes denormalized... arrrrgggghhh
- FP adder actually has more than three steps...
  - Align exponents
  - Add/subtract significands
  - Re-normalize
  - **Round**
  - **Potentially re-normalize again**
  - **Potentially round again**

- Latency in cycles of common arithmetic operations
- Source: *Software Optimization Guide for AMD Family 10h Processors, Dec 2007*
  - Intel "Core 2" chips similar

	Int 32	Int 64	Fp 32	Fp 64
Add/Subtract	1	1	4	4
Multiply	3	5	4	4
Divide	14 to 40	23 to 87	16	20

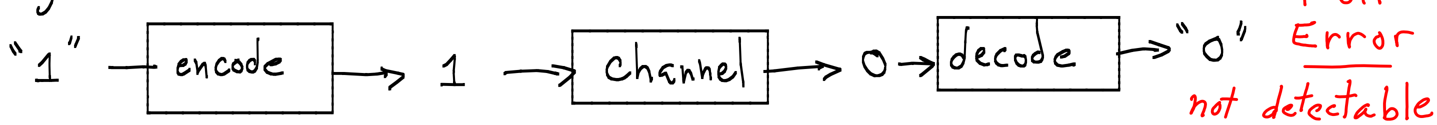
- Floating point divide faster than integer divide?
  - Why?

So much for encoding data. We could go on to audio, video, ... But, back to noise and errors.

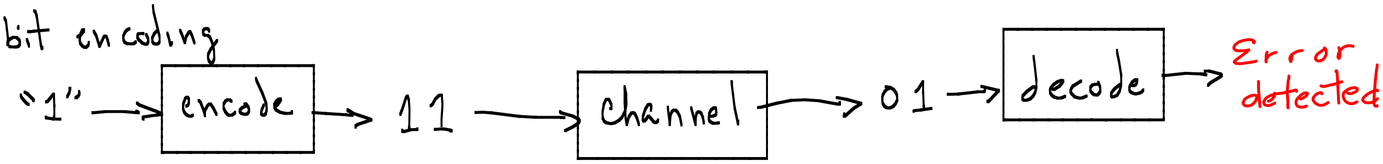
Error Detection / Correction

"message" could be a bit, a string of bits, a character, a page of characters, ...

message coded in bits:



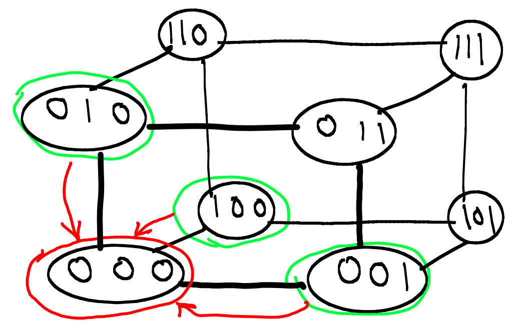
2-bit encoding



Code words 00 and 11 are good data, 10 and 01 indicate 1-bit errors. Last bit is "parity" bit, odd parity codeword indicates error. Works for k-bit messages w/ 1 parity bit (if 2-bit errors very unlikely).

Parity scheme | data | p | parity bit

1-bit Error Correction w/ 3-bit code words:
"0" ==> 000
"1" ==> 111



001 ==> "0"
010 ==> "0"
100 ==> "0"

011 ==> "1"
101 ==> "1"
110 ==> "1"

| k-bits | p-bits |
data parity

p-bit parity check

1-bit Correction, 2-bit Detection

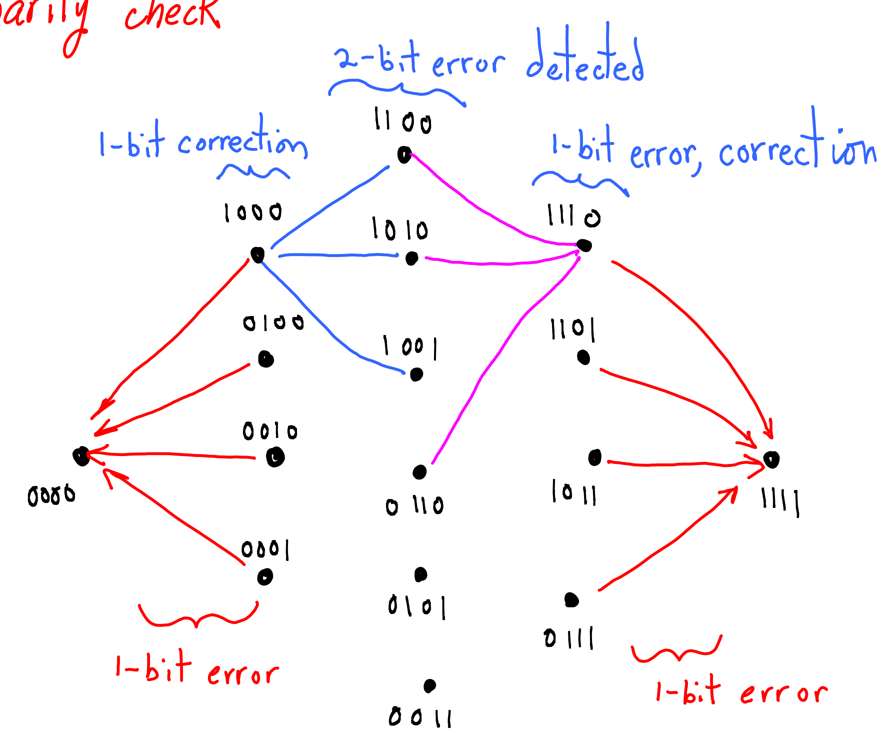
-- odd parity: 1-bit error corrected

-- exactly two 1's: 2-bit error detected

-- otherwise: no error

How many extra bits, at minimum?
Depends on noise in channel:
Shannon Noisey Coding Theorem.

We use 4 bits, 1-bit data.

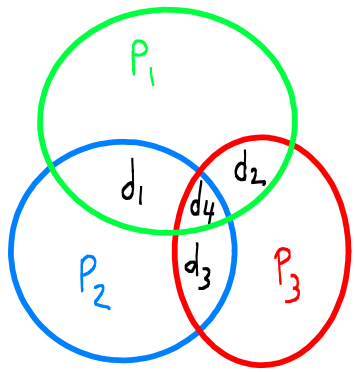


Hamming (7,4) code (Single Error Detection, Single Error Correction)

7 bits per code word

4 data bits

3 parity bits



$d_1\, d_2\, d_3\, d_4\ p_1\, p_2\, p_3$

$p_1 = \text{parity}(d_1\, d_2\, d_4)$

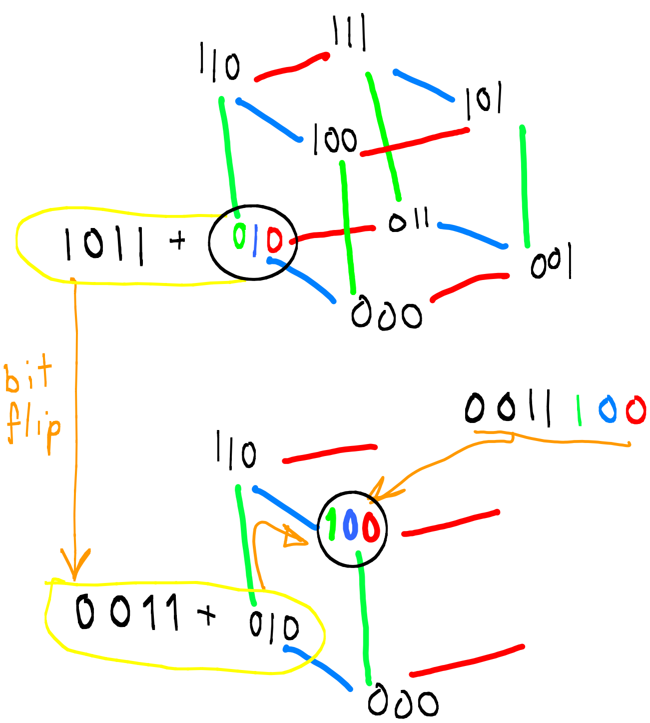
$p_2 = \text{parity}(d_1\, d_3\, d_4)$

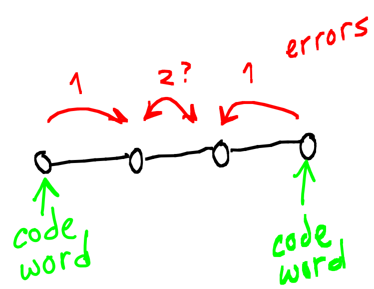
$p_3 = \text{parity}(d_2\, d_3\, d_4)$

3 steps to next codeword

1-bit error: can detect and correct

2-bit error: cannot detect





other neighbor code words

0001 111

0010 011

0111 001

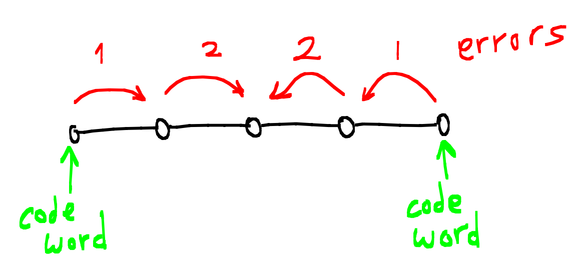
What can we do about 2-bit errors?
Add another parity bit.

$$p_4 = \text{parity}(d_1\, d_2\, d_3\, d_4\, p_1\, p_2\, p_3)$$

$$\text{code word} = d_1\, d_2\, d_3\, d_4\, p_1\, p_2\, p_3\, p_4$$

$\Longrightarrow$ 4 steps min to next code word

1-bit error: detect + correct
2-bit error: detect



0 0 0 0 0 0 0

Hamming 7,4 code:
Find distances to all other code words.
GREEN-PARITY: Bits[3, 2, 0]
BLUE-PARITY: Bits[3, 1, 0]
RED-PARITY: Bits[2, 1, 0]

0 0 0 1 1 1 1

(1-bit flip causes 3 other flips
Hamming distance
= 4)

0 0 1 0 0 1 1

Hamming distance
= 3

0 1 0 0 1 0 1

Hamming distance
= 3

1 0 0 0 1 1 0

Hamming distance
= 3

0 0 1 1 1 0 0

HD = 3

0 1 0 1 0 1 0

HD = 3

1 0 0 1 0 0 1

HD = 3

0 1 1 0 1 1 0

HD = 4

1 0 1 0 1 0 1

HD = 4

1 1 0 0 0 1 1

HD = 4

0 1 1 1 0 0 1

HD = 4

1 0 1 1 0 1 0

HD = 4

1 1 0 1 1 0 0

HD = 4

1 1 1 0 0 0 0

HD = 3

1 1 1 1 1 1 1

HD = 7

ASCII (See back cover of PP)

HEX CODE	MEANING	Printable?
00	NUL	no
01	SOH	no
...	...	
20	space	yes
...	...	
30	"0"	yes
31	"1"	yes
...	...	
41	"A"	yes
42	"B"	yes
...	...	
61	"a"	yes
62	"b"	yes
...	...	
7A	"z"	yes
...	...	
(other stuff, non-standard)		

data communications control signals

who's on first?

6D → 'm'　41 → 'A'　2F → '/'　32 → '2'

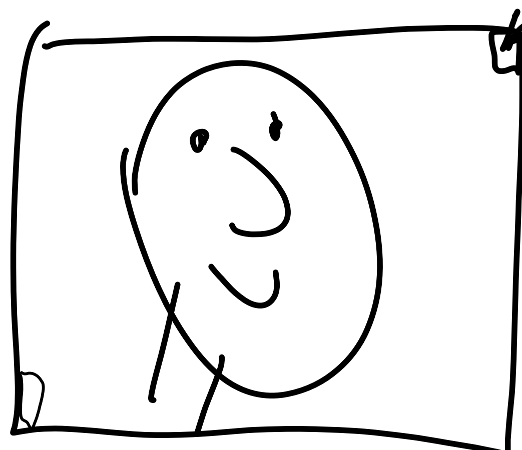
Byte Addressable

addr	Memory bits		print order
0	0011 0010 (low bit)	→ 32 ("2")	↓
1	0010 1111	→ 2F ("/")	
2	0100 0001	→ 41 ("A")	
3	0110 1101	→ 6D ("m")	

high bit

What to Print	Starting Memory Address	What is displayed (left-to-right)
4-byte number (in hex notation)	0	6D412F32
two 2-byte numbers (in hex)	0	2F32 6D41
four 1-byte numbers (in hex)	0	32 2F 41 6D
one 4-byte string	0	2 / A m

(see "od" in unix)



(8, 8, 8)
r, g, b

~~1M~~
1K
2^{10}

24 Mb
3 MB

2^{10} 1K ~~1M~~

$2^{10} \cdot 2^{10} = 2^{20}$ M

$1024 \cdot 1024 \approx 10^6$

run length