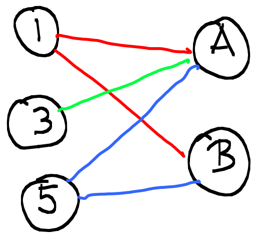
Logic

We need functions for next-state and output symbols. Since we know we can use {0,1} as a symbol basis for both states and symbols, we'll start with Boolean functions.

Set = collection of things: $\{5, 7, 9, 13\}$

Relation = pairs of things from 2 sets: $S_1 = \{1, 3, 5\}$
(binary) $S_2 = \{A, B\}$

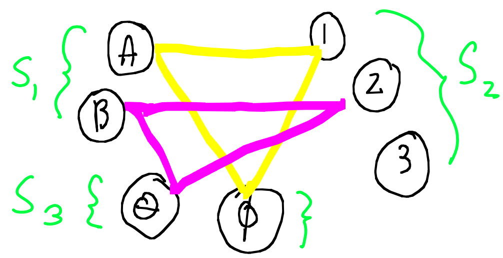


$R = \{ \{1, A\}, \{3, A\}, \{5, A\}, \{1, B\}, \{5, B\} \}$

R is a set of "tuples"

(Ternary) $S_1 = \{A, B\}$ $S_2 = \{1, 2, 3\}$ $S_3 = \{\ominus, \phi\}$

$R = \{ \{A, 1, \phi\}, \{B, 2, \ominus\}, \{A, 2, \phi\}, \{A, 3, \ominus\} \}$



Function (Map)

$f: S_1 \rightarrow S_2$

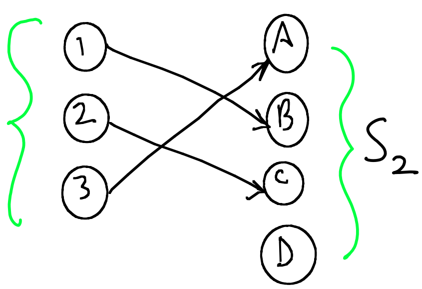
S1 = { 1, 2, 3 }
S2 = { A, B, C, D }

f = { (1,B), (2,C), (3,A) }

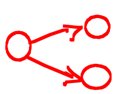
f(1) = B, f(2) = C, f(3) = A

function?
1-To-1 ?
onto ?
Operator?

exactly one arrow each



No



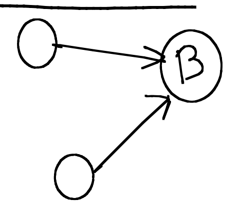
f() is a subset of crossproduct S1 X S2
(As is any relation.)

$S_1 \backslash S_2$	A	B	C	D
1	(1,A)	(1,B)	(1,C)	(1,D)
2	(2,A)	(2,B)	(2,C)	(2,D)
3	(3,A)	(3,B)	(3,C)	(3,D)

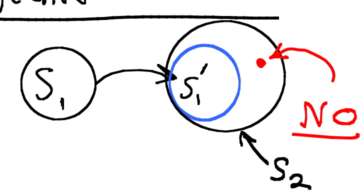
$S_1 \times S_2$

injective (1-to-1):

This doesn't happen



surjective (ON-TO):



$S_1' = f(S_1) = S_2$
image of S_1

Operator (binary)

$f: S_1 \times S_1 \rightarrow S_1$

$\boxed{+(2,3) \rightarrow 5}$

Boolean variable

$x \in \{0, 1\}$

$0 = \text{"false"} \qquad 1 = \text{"True"}$

f, a boolean n-ary operator, is a subset of all pairs (x,y):
---- x is an binary n-tuple,
---- y is binary, i.e., 0 or 1.

f is a subset of { binary n-tuples } X { 0, 1 }

Boolean function (n-ary operator)

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

n-way X product ⇒ n tuples. E.g., 4-tuples (0,1,0,0)

$$\{0,1\}^2 \begin{cases} (0,0) \longrightarrow 1 \\ (0,1) \longrightarrow 0 \\ (1,0) \longrightarrow 0 \\ (1,1) \longrightarrow 1 \end{cases}$$

variable

Value of variable

A	B	f(A,B)	
0	0	1	$f(0,0)=1$
0	1	0	$f(0,1)=0$
1	0	0	$f(1,0)=0$
1	1	1	$f(1,1)=1$

S = { (0,0), (0,1), (1,0), (1,1) } is the set of all binary 2-tuples.
f = { ((0,0), **1**), ((0,1), **0**), ((1,0), **0**), ((1,1), **1**) } is a subset of all possible pairs, S X {0, 1}.
f is a function; so, there must be exactly 4 pairs in *f*. How many pairs in S X {0, 1}? How many different functions are possible? That is, how many different 4-element subsets of S X {0, 1} are there?

We will start from here, that is, from the simplest functions we can imagine, and see if we can build on this to construct arbitrary functions. But first, we'll see what the range is we available now.

Unary boolean functions: How many?

X	f(x)
0	0
1	0

X	G(x)
0	1
1	1

X	h(x)
0	0
1	1

X	K(X)
0	1
1	0

$f : \{0,1\} \to 0 \qquad G : \{0,1\} \to 1 \qquad h(x) = X \qquad K(x) = \mathrm{NOT}(X) = \overline{X}$

Binary boolean functions: How many?

X Y	
0 0	0
0 1	0
1 0	0
1 1	0

X Y	**AND**
0 0	0
0 1	0
1 0	0
1 1	1

X Y	
0 0	0
0 1	0
1 0	1
1 1	0

X Y	
0 0	0
0 1	0
1 0	1
1 1	1

X Y	
0 0	0
0 1	1
1 0	0
1 1	0

X Y	
0 0	0
0 1	1
1 0	0
1 1	1

X Y	**XOR**
0 0	0
0 1	1
1 0	1
1 1	0

X Y	**OR**
0 0	0
0 1	1
1 0	1
1 1	1

X Y	$\overline{\textbf{OR}}$
0 0	1
0 1	0
1 0	0
1 1	0

X Y	$\overline{\textbf{XOR}}$
0 0	1
0 1	0
1 0	0
1 1	1

X Y	
0 0	1
0 1	0
1 0	1
1 1	0

X Y	
0 0	1
0 1	0
1 0	1
1 1	1

X Y	
0 0	1
0 1	1
1 0	0
1 1	0

X Y	
0 0	1
0 1	1
1 0	0
1 1	1

X Y	$\overline{\textbf{AND}}$
0 0	1
0 1	1
1 0	1
1 1	0

X Y	
0 0	1
0 1	1
1 0	1
1 1	1

How many k-ary functions are there?

e.g. 3-ary

			f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

How many ways to form this column?

Each 8-vector represents the output of a unique 3-input boolean function.

rotate

[0,0,0,0,0,0,0,0]
[0,0,0,0,0,0,0,1]
[0,0,0,0,0,0,1,0]
...
[1,1,1,1,1,1,1,1]

all possible 3-ary function output vectors

Hmm, counting in binary from 0 to (2^8 - 1).

==> 2^(2^3) different 3-ary functions.

==> 2^(2^k) different k-ary functions.

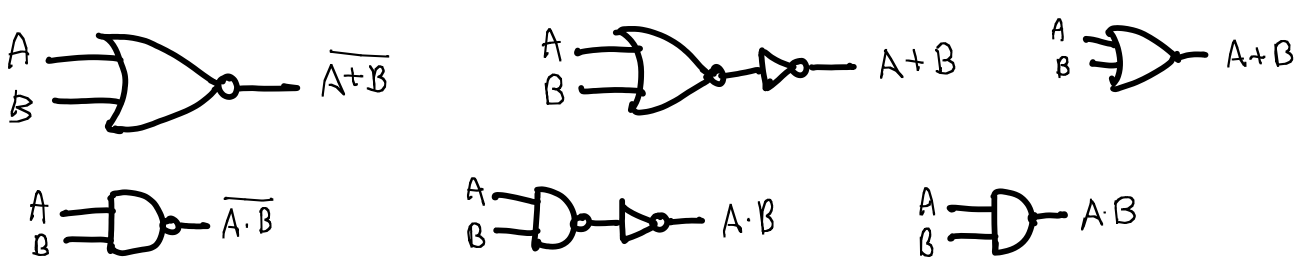
Back to our task:

We want to be able to build any arbitrary boolean function.
Why?
Because we want to build a computer, a UTM.
And why is that relevant?
UTMs are TMs, which have FSMs in them.

To be able to build any arbitrary FSM, we need to be able to build:

--- arbitrary next-state functions,
--- arbitrary output functions,
--- STATE elements.

what we can build so far:



NOT, NOR, NAND, OR, AND. Questions: Do we know how to build,

(1) EVERY k-input, n-output boolean function? (n-output = n output columns. Seems hard.)

(2) EVERY 2-input, 1-output boolean function? (Hmm, still unclear, try something easier.)

(3) EVERY 2-input, 1-output function whose output has exaclty one 1?

Hmm, none of these seem easy. The last one seems easiest. Maybe we should explore Boolean functions a bit more first.

Propositions and Logical Connectives

x	y	OR
0	0	0
0	1	1
1	0	1
1	1	1

This truth table makes the following statement:

(X OR Y) is TRUE, exactly when

(NOT(X) **and** Y) is TRUE
OR when
(X **and** NOT(Y)) is TRUE
OR when
(X **and** Y) is TRUE

X = "**It is raining**" **Y** = "**My hat is lost**"

(X + Y) is **TRUE** exactly when

(("**It is raining**" is FALSE) **AND** ("**My hat is lost**" is TRUE) **)** is a **TRUE statement**

OR

(("**It is raining**" is TRUE) **AND** ("**My hat is lost**" is FALSE) **)** is a **TRUE statement**

OR

(("**It is raining**" is TRUE) **AND** "(**My hat is lost**" is TRUE) **)** is a **TRUE statement**

x	y	AND
0	0	0
0	1	0
1	0	0
1	1	1

(X AND Y) is TRUE only once, exactly when

("**It is raining**" is TRUE) **AND** ("**My hat is lost**" is TRUE)

is a **TRUE statement**

x	y	f
0	0	1
0	1	0
1	0	0
1	1	0

Interpreting the truth table (1st row):

f (0, 0) = TRUE has two parts,

--- identify **the ROW**:

(("**It is NOT raining**" = TRUE) **AND** ("**My hat is NOT lost**" = TRUE)) is true

--- define **the function's VALUE** for THAT row:

f ("**It is raining**", "**My hat is lost**") = **TRUE**

"**It is raining**" and "**My hat is lost**" are **Boolean propositional arguments**
Each has the value TRUE
or the value FALSE.

Function Composition

$f: \mathbb{R} \to \mathbb{R} \qquad f(x) = 2x$

$g: \mathbb{R} \to \mathbb{R} \qquad g(x) = \frac{1}{3}x$

$$f(g(x)) = f(x/3) = 2\frac{x}{3}$$

$$f \circ g\,(x) \qquad f \circ g : \mathbb{R} \xrightarrow[g]{1/3} \mathbb{R} \xrightarrow[f]{2} \mathbb{R}$$

---

argument

argument

X	NOT(X)	NOT(NOT(X))
0	1	0
1	0	1

X	NOT(X)
0	1
1	0

X	NOT(NOT(X))
0	0
1	1

$$\text{AND}(\text{OR}(x,y), \text{NOT}(x)) = (x+y)\cdot\overline{x}$$

x	y	$(x+y)$	$\overline{x}$	AND($(x+y)$, $\overline{x}$)
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	1	0	0

duplicate rows?

All possible inputs are listed:
x = 0 , y = 0
x = 0 , y = 1
x = 1 , y = 0
x = 1 , y = 1

a b

a	b	a·b
0	0	0
0	1	0
1	0	0
1	1	1

Is there a row missing from AND table?

function decomposition

We can **compose** simple functions.

Decompose to simple functions?

x	y	f
0	0	0
0	1	1
1	0	0
1	1	1

x	y	g
0	0	0
0	1	1
1	0	0
1	1	0

x	y	h
0	0	0
0	1	0
1	0	0
1	1	1

Simple functions:

Exactly one output is 1.

x	y	g	h	g+h = f
0	0	0	0	0
0	1	1	0	1
1	0	0	0	0
1	1	0	1	1

f is **TRUE** exactly when

g is **TRUE**
OR
h is **TRUE**

ie., $f = (g + h)$

$$f(x,y) = \mathrm{OR}(g(x,y), h(x,y))$$

g is TRUE only once, but then h is FALSE
h is TRUE only once, but then g is FALSE

They completely and disjointly define f

SIMPLE function

is **TRUE** for
EXACTLY one input.

"single-1-output function"

Works for any 2-input function!

Decompose f ===> { set of SIMPLE functions }
OR them ===> f

We can
--- build NOT
--- build four 2-input functions (NOR, NAND, AND, OR)
--- decompose any 2-input function

Can we build every SIMPLE 2-input function?

If so, we can build ANY 2-input function.