Compute: follow a fixed procedure and produce an answer (halt), aka, **algorithm**.

What can be computed? What cannot? What can be efficiently computed (and how)?

If a single question **really is answerable "yes" or "no"**, then one of the machines, M_yes or M_no, computes the answer. We just don't know which one is correct.

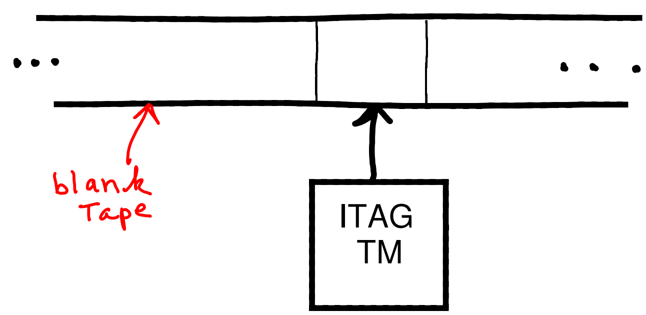
Any finite set of examples can be computed: just **make a table and look up the answer**. Just because you don't know how doesn't mean it can't be done.

Are all programs algorithms? No.

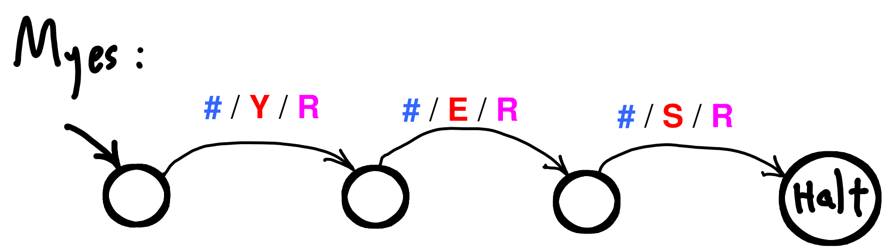
```
for (i = 1; i > 0; i = 1){
    j = j+1;
}
```

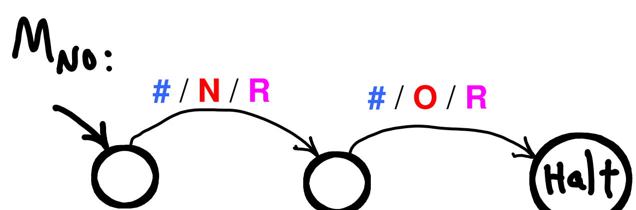
Q. Are all TMs algorithms?

"Is There A God" machine: Prints answer and halts.

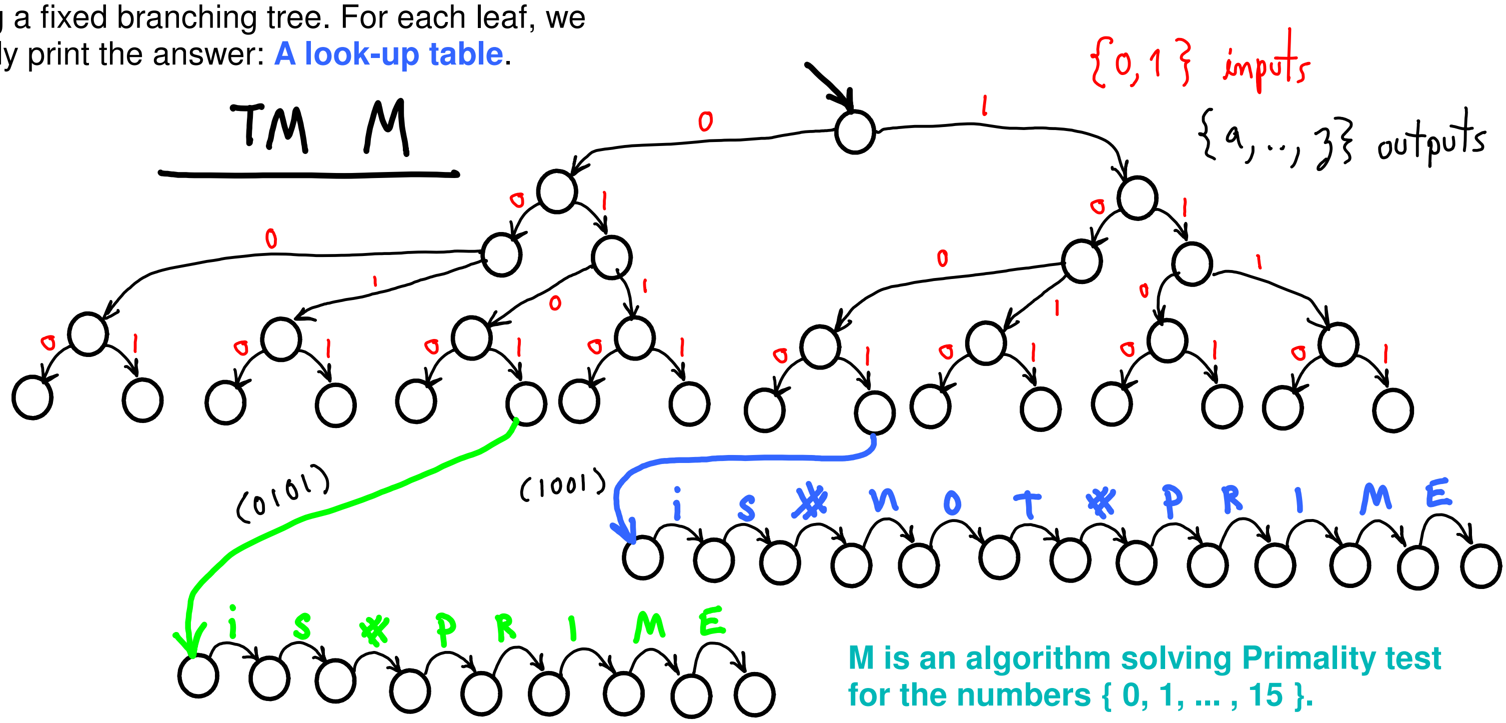


Does such a machine exist? Is the question computable?





We can **decode any finite set of questions** using a fixed branching tree. For each leaf, we simply print the answer: **A look-up table**.



M is an algorithm solving Primality test for the numbers { 0, 1, ... , 15 }.

Computability (aka, recursive)

Fermat's Last Theorem

There are no solutions to,

$$x^n + y^n = z^n$$

where $n, x, y, and\ z$ are positive integers and $n > 2$.

(Proved in 1995: Frey, Ribet, Wiles, and Taylor.)

Suppose we didn't know whether it was true or not.

Suppose we asked, Is the question,

"Is Fermat's Last Theorem true?" a computable question?

$\Longrightarrow$ Of course. Use M_{yes} or M_{no}.

Supposed we asked if this is computable?

Given some positive integer $n > 2$, is there a solution to,

$$x^n + y^n = z^n$$

where $x, y, and\ z$ are positive integers?

If Fermat's Last Theorem is true, then

M_{no} will work w/o modification. $\Rightarrow$ is computable.

Suppose it weren't true? That is, there are sol'ns for some n, but not all n.

How would we go about computing the answer?

```
FLT( n )
   pick next ( x, y, z )
   check whether  x^n + y^n == z^n
   if yes, print "there is a sol'n" ,  halt;
   else, repeat
```

This works when there is a sol'n for a particular n.

?!? Will it halt? For every n?

Diagonalization

How many questions are there? How many TMs?

In our encoding, we used a string of 0s and 1s to represent a TM. Symbol set is {0, 1}.

--- **Each TM** can be identified with **an integer**. (There are infinitely many machines that do the same thing.)

--- **Each input tape** configuration can be identified with **an integer**.

--- **Each output tape** configuration can be identified with **an integer**.

--- A **TM** can be looked at as **an integer function**: given input, **x**, machine **M** produces integer **M(x)**.
---(**M** might loop forever on some inputs; if so, then **M** is a "partial" function.)

List all TM functions

unary encoding of a TM is a binary number:

100100101101101011001010101011100110110110101001101011011001110000

input

an integer, encoded input tape ...000...010 2

TM	0	1	2	3	4 ...
M_0 :	$M_0(0)$	$M_0(1)$	$M_0(2)$	$M_0(3)$	...
M_1 :	$M_1(0)$	$M_1(1)$	$M_1(2)$	∞	...
M_2 :	$M_2(0)$	$M_2(1)$	...		
M_3 :	...				

← all outputs for $M_\emptyset$

← all outputs for M_1

M_1 loops forever when started w/ tape configuration 3.

an integer, encoded description of M_3

an integer, encoded tape when M_2 halts on input 1.

Q. Can you **encode** an **arbitrary input tape**, in a **arbitrary symbol set**, using **only {0, 1}?** Hint, use unary encoding.

(Recall, only a finite portion of tape is non-blank.)
That binary string is an integer.

Computable (real) numbers:
Given e, output finite number of digits of x so that the output is within e of x.
π is such a number.

How many integer functions are there?

Consider a function *g*( ), which we describe by saying that *g*( ) **is different** from **all the functions** in the list above. How? Because *g*( ) is,

not the same as M_0 for the **first output, g(0)**,
and it is,
not the same as M_1 for the **second output, g(1)**,
and it is,
...

--- **Diagonalization:**

g(0) **!= M_0(0)**
g(1) **!= M_1(1)**
g(2) **!= M_2(2)**
... (forever)

---- *g*( ) is **different from every function** in the list; so,
g( ) is **not in the list**!

--- How many different ways are there to pick g()?
g( ) is any element from N - { M_0(0) }
g( ) is any element from N - { M_1(1) }
g( ) is any element from N - { M_2(2) }
...

There are so many different functions, ***g*( ),** proportionally, that the **probability of randomly** picking a function from **a bag of integer functions** and having that **function** correspond to **some TM is 0**.

[What the heck does that really mean?]

That is,

There are a lot of functions (more than all the positive integers).

Nearly all are incomputable

input

	0	1	2	...
M_0	$\neq$	$M_0(1)$	$M_0(2)$	...
M_1	$M_1(0)$	$\neq$	$M_1(2)$	...
M_2	$M_2(0)$	$M_2(1)$	$\neq$	...
...				$\ddots$

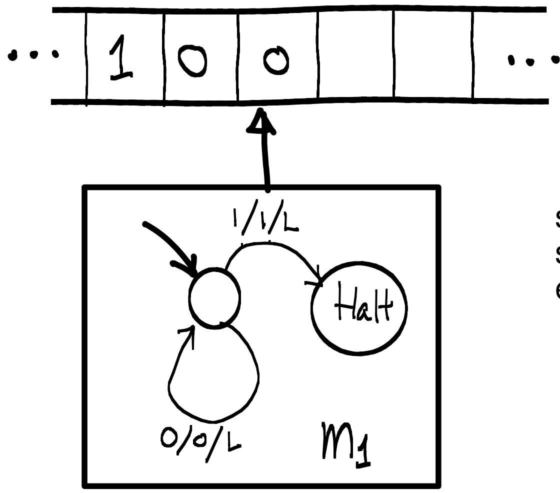
Maybe it only means we don't know how to arrange an infinite list of TMs? We are limited in our own computing power?

How "numerous" is "infinity to the infinity"?

How can we know we are able to produce g() this way?

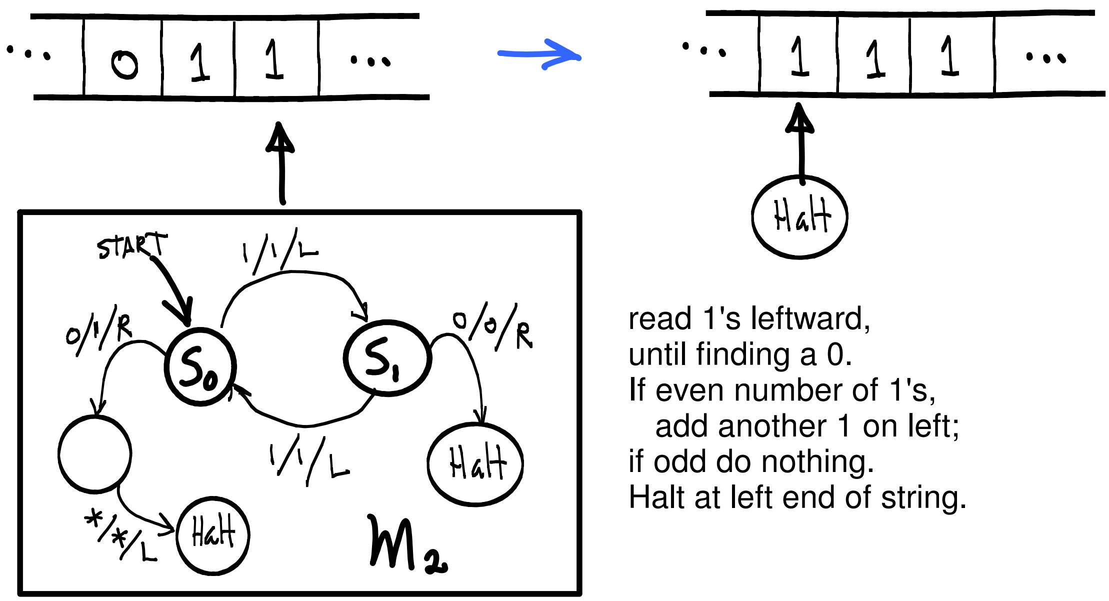
As long as we are building TMs, lets see how to simplify our work.
How about combining two TMs to make a new one?

"spin left ϕ_s"



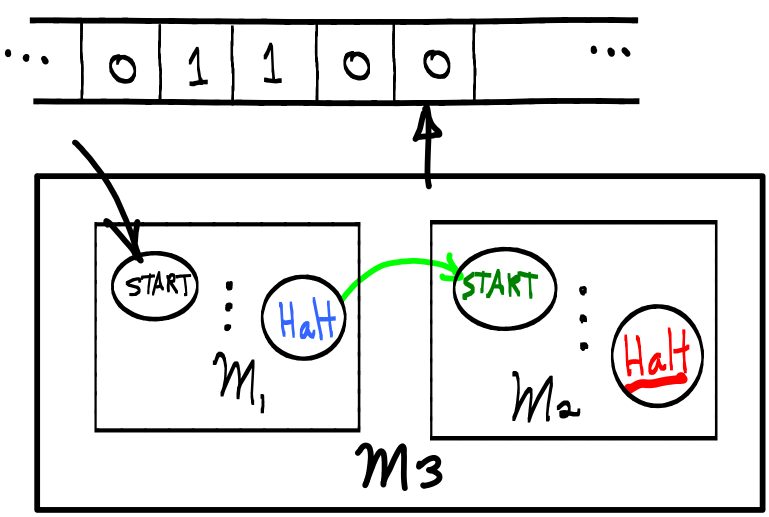
skip 0's to the left,
stop at the first 1,
end up w/ R/W to its left.

"Parity fix"



read 1's leftward,
until finding a 0.
If even number of 1's,
add another 1 on left;
if odd do nothing.
Halt at left end of string.

"Spin-left ϕ_s"
Then
"parity fix"



M3 starts in M1's start state.

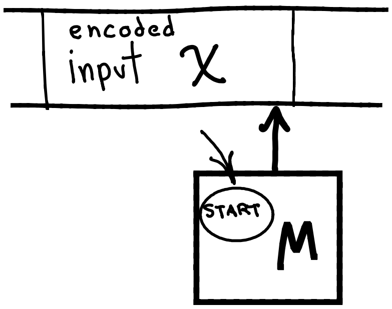
- Every M1 state transistion that goes to **M1**'s "**HALT**" state is instead connected to **M2**'s **START** state.

- **M3**'s halting state is **M2**'s "**HALT**" state.

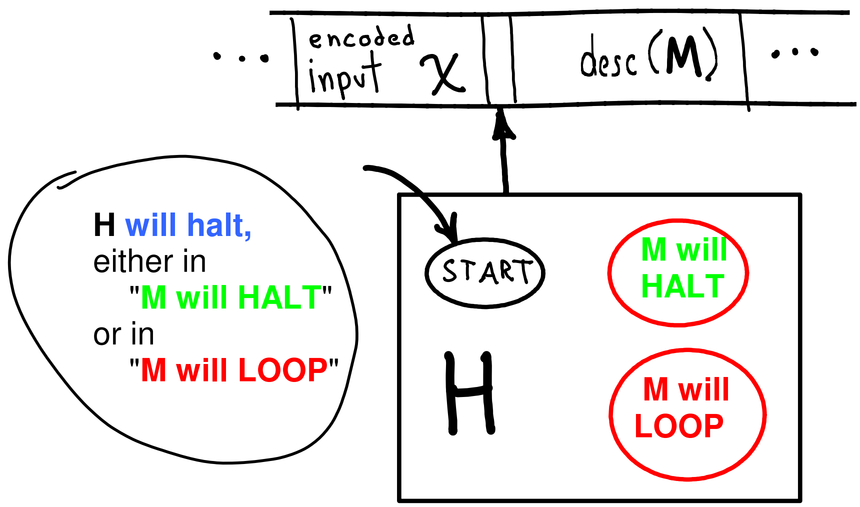
Lemma:
All TM's with x as input, either (1) HALT or (2) LOOP FOREVER. (exercise: prove the lemma.)

A very special integer function: The **Halting function**:

input:	integer **xM**	(**xM** == an **encoding of input x followed by an encoding of M.**)
output:	"**1**" if **xM** HALTS; "**0**" otherwise.	(**xM** == **M reading x as its input.**)



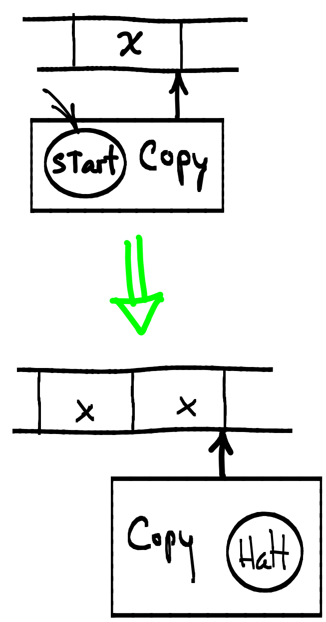
Question:
Will M eventually halt,
or loop forever?

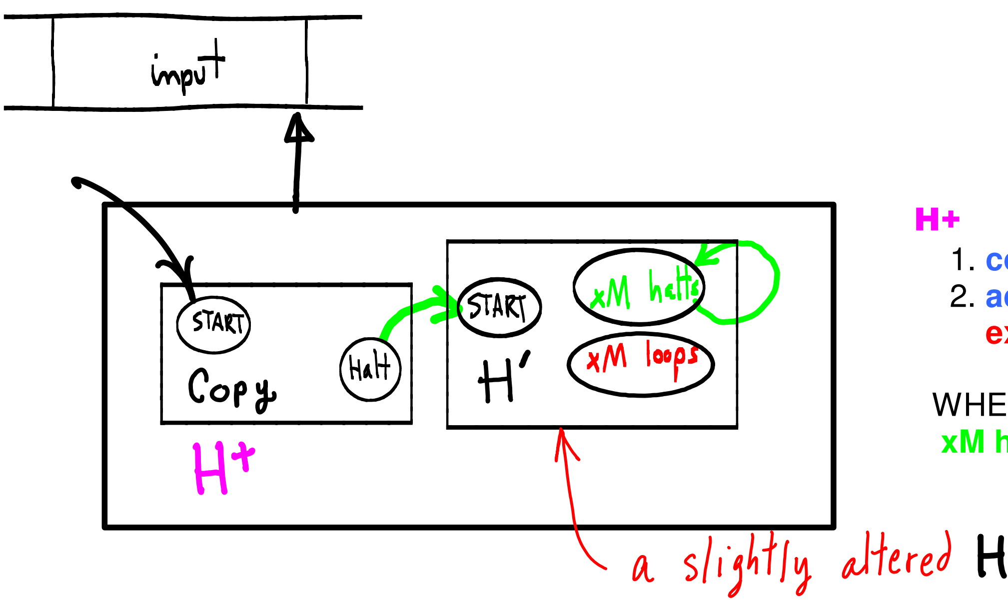


Q. Can there be such an H function. Is it possible?
Asummption: Either (**H exists**) IS TRUE, or (**H does not exist**) IS TRUE.

Suppose (**H exists**) **IS TRUE**.

Then we can **build another machine**, **H+**, using **H** and a "**Copy**" TM.

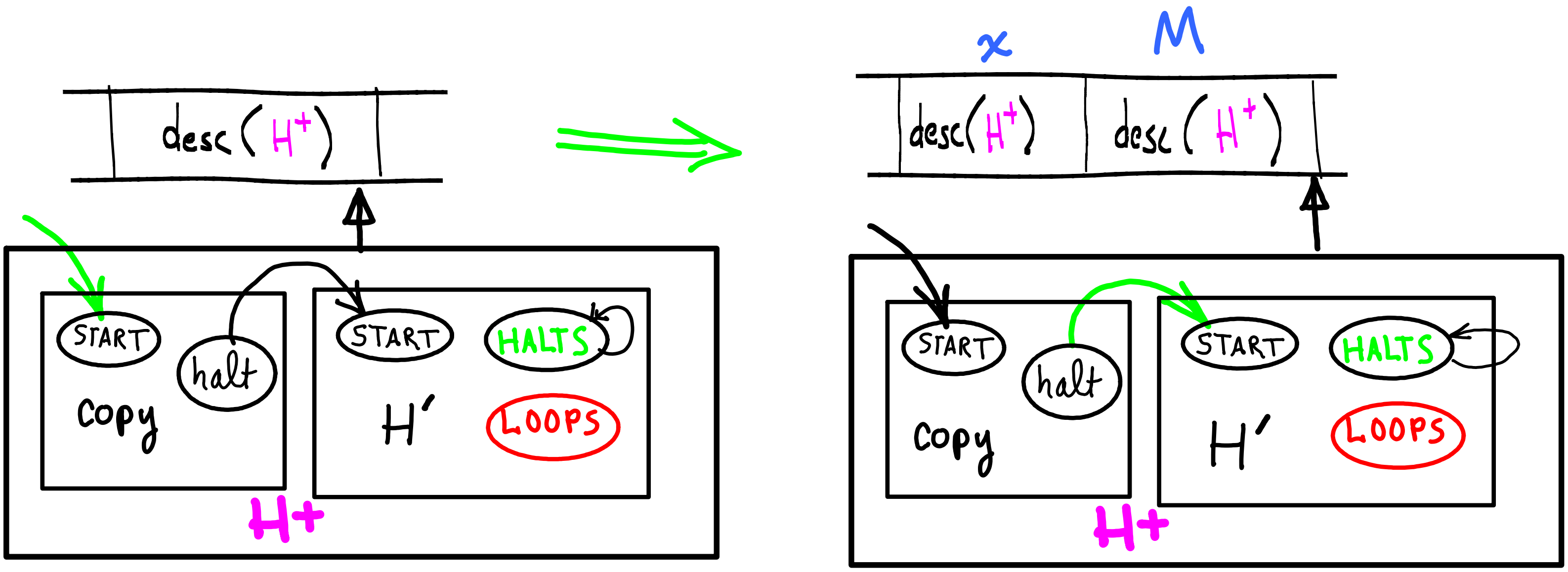




H+
1. **copies its input.**
2. **acts as H would,**
   except:

WHEN **H+** reaches
xM halts, **H+ LOOPS.**

Consider putting **desc(H+) on H+'s input tape. What must happen?**

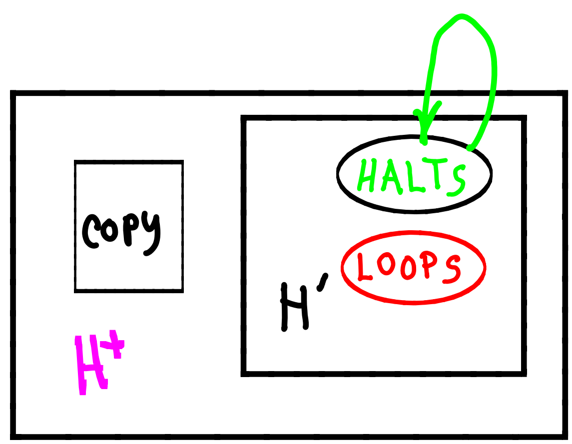


H+ first does exactly what **Copy** would do, **copy its input**.
Next, **H+ acts exactly as H would**.

The **tape is now** thought of as,
an **input**, **x = desc(H+)**, **followed by a machine description**, **desc(M)** = **desc(H+)**.

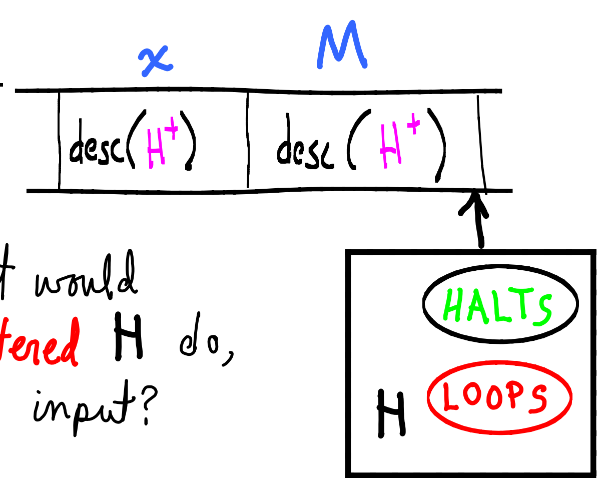
H+ WILL either (because **H** always halts in HALTS or LOOPS)
(**reach HALTS and then loop**)
OR
(**reach LOOPS and then halt**).

SUPPOSE H+ loops.



1. **H+ reached HALTS**.
2. Then **H** with input **xM** == **desc(H+) desc(H+)**, **would have halted** in **HALTS**.
3. BUT **H+ reading desc(H+) loops (our assumption)**.
4. Since **H is correct**, **it would not go to HALTS**.
5. **H+ cannot reach HALTS, and does not loop.**
6. This **contradicts our assumption** that **H+ loops**.

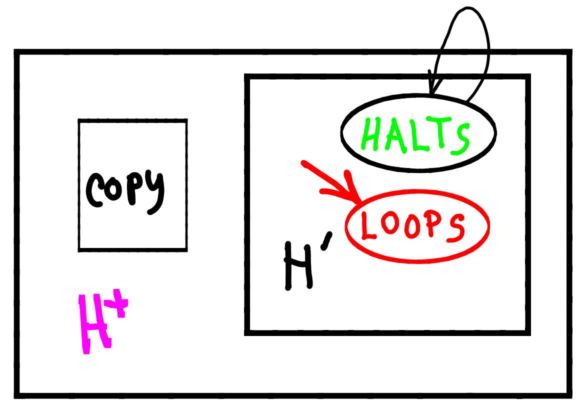
We **assumed H exists, i.e., it works correctly**.
Assuming also that H+ loops leads to a contradiction.
At least **one of these assumptions** must be **false**.



ask, what would the unaltered H do, given this input?

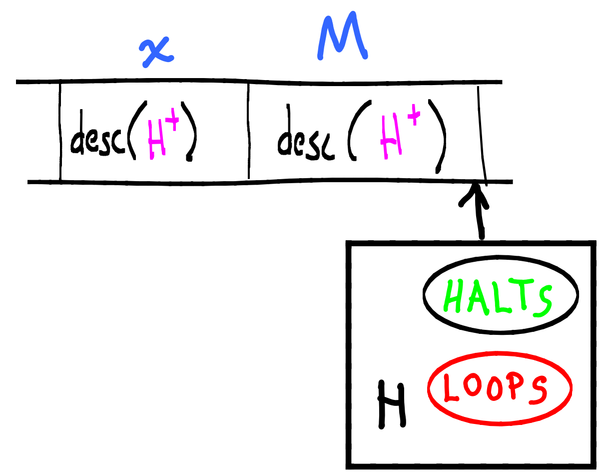
SUPPOSE H+ halts.

1. **H+ reached LOOPS.**
2. **H reading desc(H+) desc(H+) must reach LOOPS.**
3. BUT **desc(H+) H+ halts.**
4. **H is correct**; so, **H cannot reach LOOPS.**
5. **desc(H+) H+ cannot reach LOOPS.**



We assumed H is correct.
Assuming also that H+ halts leads to a contradiction.

If H exists, H+ exists, is a TM, and either halts or loops. (Building H+ from H was easy and resulted in a TM.)



But both cases (H+ either halts or loops) lead to contradictions.

The assumption that H exists must be false.

This is better than diagonalization: we have a real, uncomputable function. The function exists because every TM M either halts of loops forever, give an input x.

There is a function H() mapping
H: { xM } ===> {0, 1}
from positive integers to {0, 1}, but no TM can compute it.

Are we doomed?

Build something H- that partially computes the Halting Problem?

Works for some inputs, but not others?

Works for some fixed number of inputs?

Has a lookup table?

How many machines act exactly like any given description?

How many descriptions are there?

How many other things are not Turing computable? What does this say about cognition? ...???

Another Method? Does it work? Why?

```
Hnew( x, M)

    print "loops forever"

    1. Simulate xM for one step.
    2. If  xM halted
           print "halts"
       else
           go to 1.
```

$\Rightarrow$ Is HP computable?

Bottom Line

Suppose we try to write a program $H(x, M)$.

We **succeed** for some special cases $\{M_1, M_{25}, M_{300}, \ldots\}$

But, we always find a new M_i and have to rewrite $H(x,M)$. Also, we get it to work for $\{x_1, x_2, \ldots\}$, but find a new x_i for which $x_i M_j$ loops (or halts) (can we figure that out?), and our $H(x_i M_j)$ says it will loop. Back to re-writing our H().

HP $\Rightarrow$ We will never be bored!

Formal Proof

Notation: "[halts]" means "H+ halts when reading its own description"; "[loops]" is to be read similarly; "==>" means, "implies", in the logical sense of material implication; "-" means logical NOT.

1. (H exists) ==> (H+ exists (is a TM)) (by properties of TM)

2. (H+ exists) ==> [halts] OR [loops] (by properties of TM)

3. (H+ exists) ==> -[loops] AND -[halts] (demonstrated above)

4. (H exists) ==> ([halts] OR [loops]) AND (-[loops] AND -[halts]) (by 1. and 2.)

5. (H exists) ==> ([halts] AND -[halts]) OR ([loops] AND -[loops]) (by AND/OR properties)

6. p ==> q EQUALS -q ==> -p (by properties of "==>")

7. -(([halts] AND -[halts]) OR ([loops] AND -[loops])) ==> -(H exists) (by 5. and 6.)

8. -(([halts] AND -[halts]) OR ([loops] AND -[loops])) (true by AND/OR properties)

9. -(H exists) (syllogism applied to 7. and 8.)