Lecture 14:
Algorithms for HMMs

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(some slides from Sharon Goldwater;
thanks to Jonathan May for bug fixes)

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Recap: tagging

- POS tagging is a sequence labelling task.
- We can tackle it with a model (HMM) that uses two sources of information:
  - The word itself
  - The tags assigned to surrounding words
- The second source of information means we can't just tag each word independently.

Local Tagging

Words:

Possible tags:
(ordered by frequency for each word)

<s>	one	dog	bit	</s>
<s>	CD	NN	NN	</s>
	NN	VB	VBD	
	PRP			

- Choosing the best tag for each word independently, i.e. not considering tag context, gives the wrong answer (<s> CD NN NN </s>).
- Though NN is more frequent for ‘bit’, tagging it as VBD may yield a better *sequence* (<s> CD NN VB </s>)
  - because P(VBD|NN) and P(</s>|VBD) are high.

Recap: HMM

- Elements of HMM:
  - Set of states (tags)
  - Output alphabet (word types)
  - Start state (beginning of sentence)
  - State transition probabilities $P(t_i \mid t_{i-1})$
  - Output probabilities from each state $P(w_i \mid t_i)$

Recap: HMM

- Given a sentence $\mathbf{W}=\mathrm{w}_1\ldots\mathrm{w}_\mathrm{n}$ with tags $\mathbf{T}=\mathrm{t}_1\ldots\mathrm{t}_\mathrm{n}$, compute $\mathrm{P}(\mathbf{W},\mathbf{T})$ as:

$$P(\mathbf{W},\mathbf{T}) = \prod_{i=1}^{n} P(w_i|t_i)P(t_i|t_{i-1})$$

- But we want to find $\mathrm{argmax}_{\mathbf{T}}\, P(\mathbf{T}|\mathbf{W})$without enumerating all possible tag sequences $\mathbf{T}$
  - Use a greedy approximation, or
  - Use Viterbi algorithm to store partial computations.

Greedy Tagging

Words:

Possible tags:
(ordered by frequency for each word)

<s>	one	dog	bit	</s>
<s>	CD	NN	NN	</s>
	NN	VB	VBD	
	PRP			

- For i = 1 to N: choose the tag that maximizes
  - transition probability $P(t_i|t_{i-1}) \times$
  - emission probability $P(w_i|t_i)$
- This uses tag context but is still suboptimal. Why?
  - It commits to a tag before seeing subsequent tags.
  - It could be the case that ALL possible next tags have low transition probabilities. E.g., if a tag is unlikely to occur at the end of the sentence, that is disregarded when going left to right.

Greedy vs. Dynamic Programming

- The greedy algorithm is **fast**: we just have to make one decision per token, and we're done.
  - Runtime complexity?
  - $O(TN)$ with T tags, length-N sentence
- But subsequent words have no effect on each decision, so the result is likely to be **suboptimal**.
- Dynamic programming search gives an **optimal global** solution, but requires some bookkeeping (= more computation). Postpones decision about any tag until we can be sure it's optimal.

Viterbi Tagging: intuition

Words:

Possible tags:
(ordered by frequency for each word)

<s>	one	dog	bit	</s>
<s>	CD	NN	NN	</s>
	NN	VB	VBD	
	PRP			

- Suppose we have already computed
  a) The best tag sequence for <s> … bit that ends in NN.
  b) The best tag sequence for <s> … bit that ends in VBD.
- Then, the best full sequence would be either
  - sequence (a) extended to include </s>, or
  - sequence (b) extended to include </s>.

Viterbi Tagging: intuition

Words:

Possible tags: (ordered by frequency for each word)

<s>	one	dog	bit	</s>
<s>	CD	NN	NN	</s>
	NN	VB	VBD	
	PRP			

- But similarly, to get
  a) The best tag sequence for <s> … bit that ends in NN.
- We could extend one of:
  – The best tag sequence for <s> … dog that ends in NN.
  – The best tag sequence for <s> … dog that ends in VB.
- And so on…

Viterbi: high-level picture

- Want to find $\operatorname{argmax}_{\mathbf{T}} P(\mathbf{T}|\mathbf{W})$
- Intuition: the best path of length i ending in state t must include the best path of length i-1 to the previous state. So,
  - Find the best path of length i-1 to each state.
  - Consider extending each of those by 1 step, to state t.
  - Take the best of those options as the best path to state t.

Viterbi algorithm

- Use a **chart** to store partial results as we go
  - $\mathrm{T} \times \mathrm{N}$ table, where $v(t,i)$ is the probability* of the best state sequence for $\mathrm{w}_1 \ldots \mathrm{w}_\mathrm{i}$ that ends in state t.

*Specifically, v(t,i) stores the max of the joint probability $P(w_1 \ldots w_i, t_1 \ldots t_{i-1}, t_i = t | \lambda)$

Viterbi algorithm

- Use a **chart** to store partial results as we go
  - $\mathrm{T} \times \mathrm{N}$ table, where $v(t,i)$ is the probability* of the best state sequence for $\mathrm{w}_1 \ldots \mathrm{w}_\mathrm{i}$ that ends in state t.
- Fill in columns from left to right, with

$$v(t,i) = \max_{t'} v(t',i-1) \cdot P(t|t') \cdot P(w_i|t_i)$$

  - The max is over each possible previous tag t'
- Store a **backtrace** to show, for each cell, which state at $i-1$ we came from.

*Specifically, v(t,i) stores the max of the joint probability $P(w_1 \ldots w_i, t_1 \ldots t_{i-1}, t_i = t | \lambda)$

Transition and Output Probabilities

Transition matrix: $P(t_i \mid t_{i-1})$:

	Noun	Verb	Det	Prep	Adv	</s>
<s>	.3	.1	.3	.2	.1	0
Noun	.2	.4	.01	.3	.04	.05
Verb	.3	.05	.3	.2	.1	.05
Det	.9	.01	.01	.01	.07	0
Prep	.4	.05	.4	.1	.05	0
Adv	.1	.5	.1	.1	.1	.1

Emission matrix: $P(w_i \mid t_i)$:

	a	cat	doctor	in	is	the	very
Noun	0	.5	.4	0	.1	0	0
Verb	0	0	.1	0	.9	0	0
Det	.3	0	0	0	0	.7	0
Prep	0	0	0	1.0	0	0	0
Adv	0	0	0	.1	0	0	.9

Example

Suppose W=the doctor is in. Our initially empty table:

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun					
Verb					
Det					
Prep					
Adv					

Filling in the first column

Suppose W=the doctor is in. Our initially empty table:

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0				
Verb	0				
Det	.21				
Prep	0				
Adv	0				

$v(\text{Noun}, \text{the}) = P(\text{Noun}|<\text{s}>)P(\text{the}|\text{Noun})$=.3(0)

…

$v(\text{Det}, \text{the}) = P(\text{Det}|<\text{s}>)\ P(\text{the}|\text{Det})$=.3(.7)

The second column

$$v(\text{Noun}, \text{doctor})$$
$$= \max_{t'} v(t', \text{the}) \cdot P(\text{Noun}|t') \cdot P(\text{doctor}|\text{Noun})$$

$\boldsymbol{v}$	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	?			
Verb	0				
Det	.21				
Prep	0				
Adv	0				

$P(\text{Noun}|\text{Det})\ P(\text{doctor}|\text{Noun})$=.3(.4)

The second column

$$v(\text{Noun}, \text{doctor})$$

$$= \max_{t'} v(t', \text{the}) \cdot P(\text{Noun}|t') \cdot P(\text{doctor}|\text{Noun})$$

= max { 0, 0, .21(.36), 0, 0 } = .0756

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756			
Verb	0				
Det	.21				
Prep	0				
Adv	0				

$P(\text{Noun}|\text{Det})\ P(\text{doctor}|\text{Noun})$=.9(.4)

The second column

$v(\text{Verb}, \text{doctor})$

$$= \max_{t'} v(t', \text{the}) \cdot P(\text{Verb}|t') \cdot P(\text{doctor}|\text{Verb})$$

= max { 0, 0, .21(.001), 0, 0 } = .00021

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756			
Verb	0	.00021			
Det	.21				
Prep	0				
Adv	0				

$P(\text{Verb}|\text{Det})\ P(\text{doctor}|\text{Verb})$=.01(.1)

The second column

$v(\text{Verb}, \text{doctor})$

$$= \max_{t'} v(t', \text{the}) \cdot P(\text{Verb}|t') \cdot P(\text{doctor}|\text{Verb})$$

= max { 0, 0, .21(.001), 0, 0 } = .00021

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756			
Verb	0	.00021			
Det	.21	0			
Prep	0	0			
Adv	0	0			

$P(\text{Verb}|\text{Det})\ P(\text{doctor}|\text{Verb})$=.01(.1)

The third column

$v(\text{Noun}, \text{is})$

$$= \max_{t'} v(t', \text{doctor}) \cdot P(\text{Noun}|t') \cdot P(\text{is}|\text{Noun})$$

= max { .0756(.02), .00021(.03), 0, 0, 0 } = .001512

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512		
Verb	0	.00021			
Det	.21	0			
Prep	0	0			
Adv	0	0			

$P(\text{Noun}|\text{Noun})\ P(\text{is}|\text{Noun})$=.2(.1)=.02

$P(\text{Noun}|\text{Verb})\ P(\text{is}|\text{Noun})$=.3(.1)=.03

The third column

$v(\text{Verb}, \text{is})$

$$= \max_{t'} v(t', \text{doctor}) \cdot P(\text{Verb}|t') \cdot P(\text{is}|\text{Verb})$$

= max { .0756(.36), .00021(.045), 0, 0, 0 } = .027216

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512		
Verb	0	.00021	.027216		
Det	.21	0	0		
Prep	0	0	0		
Adv	0	0	0		

$P(\text{Verb}|\text{Noun})\ P(\text{is}|\text{Verb})$=.4(.9)=.36

$P(\text{Verb}|\text{Verb})\ P(\text{is}|\text{Verb})$=.05(.9)=.045

The fourth column

$v(\text{Prep}, \text{in})$

$$= \max_{t'} v(t', \text{is}) \cdot P(\text{Prep}|t') \cdot P(\text{in}|\text{Prep})$$

= max { .001512(.3), .027216(.2), 0, 0, 0 } = .005443

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	
Prep	0	0	0	.005443	
Adv	0	0	0		

$P(\text{Prep}|\text{Noun})\ P(\text{in}|\text{Prep})$=.3(1.0)

$P(\text{Prep}|\text{Verb})\ P(\text{in}|\text{Prep})$=.2(1.0)

The fourth column

$v(\text{Prep}, \text{in})$

$$= \max_{t'} v(t', \text{is}) \cdot P(\text{Prep}|t') \cdot P(\text{in}|\text{Prep})$$

= max { .000504(.004), .027216(.01), 0, 0, 0 } = .00027

$\boldsymbol{v}$	$\text{w}_1\text{=the}$	$\text{w}_2\text{=doctor}$	$\text{w}_3\text{=is}$	$\text{w}_4\text{=in}$	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	

$P(\text{Adv}|\text{Noun})\ P(\text{in}|\text{Adv})$=.04(.1)

$P(\text{Adv}|\text{Verb})\ P(\text{in}|\text{Adv})$=.1(.1)

End of sentence

$v(</s>)$

$= \max_{t'} v(t', \mathrm{in}) \cdot P(</s>|t')$

$= \max \{ 0, 0, 0, .005443(0), .000272(.1) \} = .0000272$

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	.0000272
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	

$P(</s>|\mathrm{Prep})=0$

$P(</s>|\mathrm{Adv})=.1$

Completed Viterbi Chart

$\boldsymbol{v}$	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	.000027 2
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	

Following the Backtraces

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	.0000272
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	

Following the Backtraces

v	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	.0000272
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	

Following the Backtraces

$\boldsymbol{v}$	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	.0000272
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	

Following the Backtraces

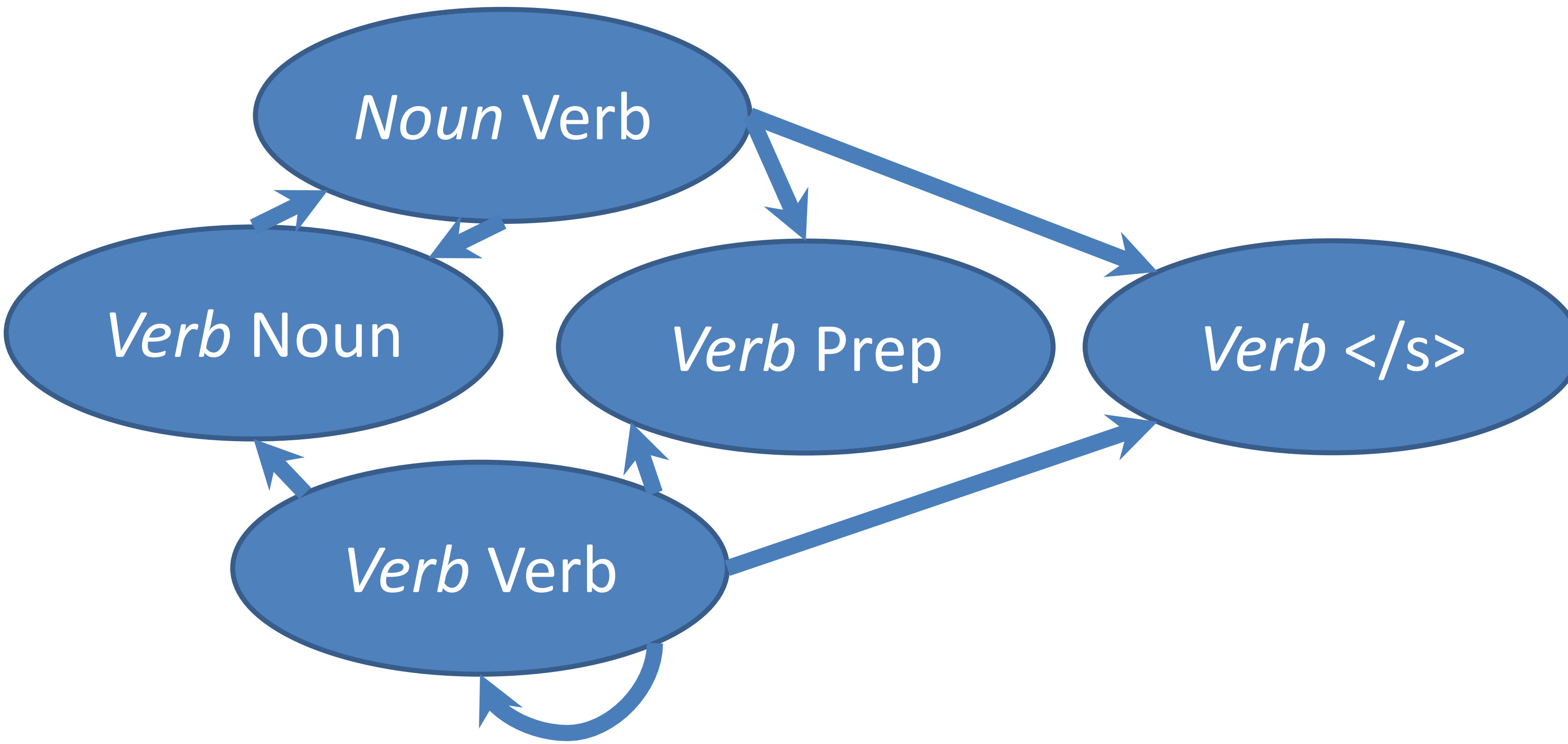
$\boldsymbol{v}$	w_1=the	w_2=doctor	w_3=is	w_4=in	</s>
Noun	0	.0756	.001512	0	
Verb	0	.00021	.027216	0	
Det	.21	0	0	0	.0000272
Prep	0	0	0	.005443	
Adv	0	0	0	.000272	
	Det	Noun	Verb	Prep	

Implementation and efficiency

- For sequence length N with T possible tags,
  - Enumeration takes $O(T^N)$ time and $O(N)$ space.
  - Bigram Viterbi takes $O(T^2N)$ time and $O(TN)$ space.
  - Viterbi is exhaustive: further speedups might be had using methods that prune the search space.
- As with N-gram models, chart probs get really tiny really fast, causing underflow.
  - So, we use **costs** (neg log probs) instead.
  - Take minimum over sum of costs, instead of maximum over product of probs.

Higher-order Viterbi

- For a tag **trigram** model with T possible tags, we effectively need T^2 states
  - n-gram Viterbi requires T^{n-1} states, takes $O(T^nN)$ time and $O(T^{n-1}N)$ space.



HMMs: what else?

- Using Viterbi, we can find the best tags for a sentence (**decoding**), and get $P(\mathbf{W}, \mathbf{T})$.
- We might also want to
  - Compute the **likelihood** $P(\mathbf{W})$, i.e., the probability of a sentence regardless of its tags (a language model!)
  - **learn** the best set of parameters (transition & emission probs.) given only an *unannotated* corpus of sentences.

Computing the likelihood

- From probability theory, we know that

$$P(\mathbf{W}) = \sum_{\mathbf{T}} P(\mathbf{W}, \mathbf{T})$$

- There are an exponential number of $\mathbf{T}$s.
- Again, by computing and storing partial results, we can solve efficiently.
- *(Advanced slides show the algorithm for those who are interested!)*

Summary

- HMM: a generative model of sentences using hidden state sequence
- Greedy tagging: fast but suboptimal
- Dynamic programming algorithms to compute
  - Best tag sequence given words (Viterbi algorithm)
  - Likelihood (forward algorithm—*see advanced slides*)
  - Best parameters from unannotated corpus (forward-backward algorithm, an instance of EM—*see advanced slides*)

Advanced Topics

(the following slides are just for people who are interested)

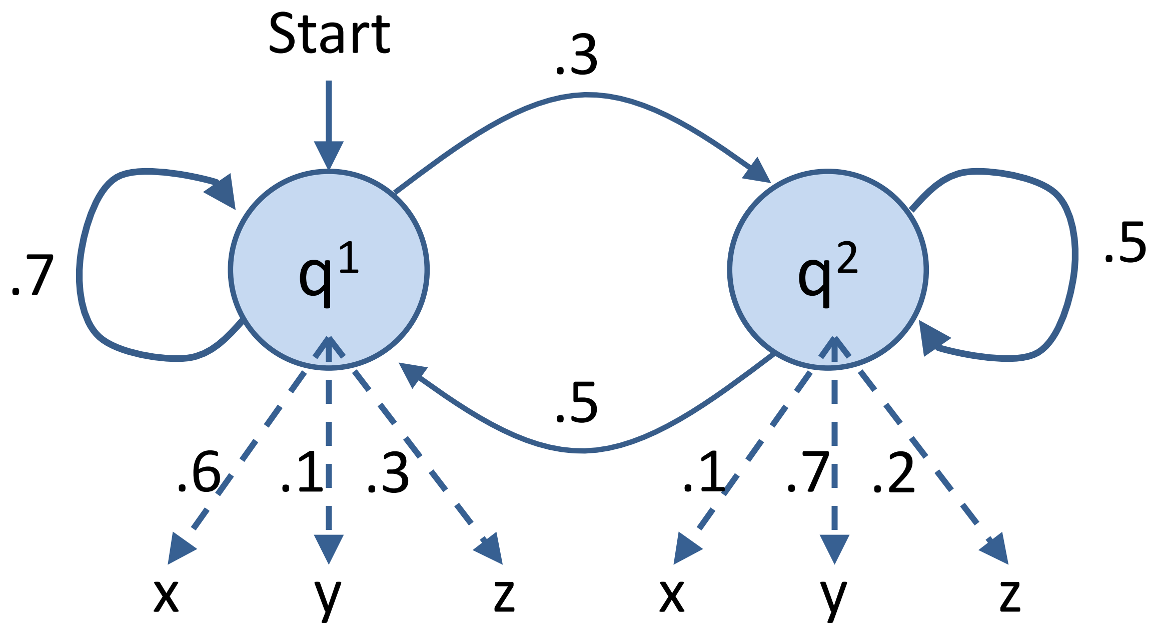
Notation

- Sequence of observations over time $o_1, o_2, \ldots, o_N$
  - here, words in sentence
- Vocabulary size V of possible observations
- Set of possible states $q^1, q^2, \ldots, q^T$ (see note next slide)
  - here, tags
- A, an $T \times T$ matrix of transition probabilities
  - a_{ij}: the prob of transitioning from state i to j.
- B, an $T \times V$ matrix of output probabilities
  - $b_i(o_t)$: the prob of emitting o_t from state i.

Note on notation

- J&M use $q_1, q_2, \ldots, q_N$ for set of states, but *also* use $q_1, q_2, \ldots, q_N$ for state sequence over time.
  - So, just seeing q_1 is ambiguous (though usually disambiguated from context).
  - I'll instead use q^i for state names, and q_n for state at time n.
  - So we could have $q_n = q^i$, meaning: the state we're in at time n is q^i.

HMM example w/ new notation



- States $\{q^1, q^2\}$ (or $\{<s>, q^1, q^2\}$): think *NN, VB*
- Output symbols $\{x, y, z\}$: think *chair, dog, help*

Adapted from Manning & Schuetze, Fig 9.2

HMM example w/ new notation

- A possible sequence of outputs for this HMM:

  z y y x y z x z z

- A possible sequence of states for this HMM:

  $q^1 \; q^2 \; q^2 \; q^1 \; q^1 \; q^2 \; q^1 \; q^1 \; q^1$

- For these examples, $N = 9$, $q_3 = q^2$ and $o_3 = y$

Transition and Output Probabilities

- Transition matrix A:

  $a_{ij} = P(q^j \mid q^i)$

  Ex: $P(q_n = q^2 \mid q_{n-1} = q^1) = .3$

	q^1	q^2
<s>	1	0
q^1	.7	.3
q^2	.5	.5

- Output matrix B:

  $b_i(o) = P(o \mid q^i)$

  Ex: $P(o_n = y \mid q_n = q^1) = .1$

	x	y	z
q^1	.6	.1	.3
q^2	.1	.7	.2

Forward algorithm

- Use a table with cells $\alpha(j,t)$: the probability of being in state j after seeing $o_1 \ldots o_t$ (**forward probability**).

$$\alpha(j,t) = P(o_1, o_2, \ldots ot, qt = j|\lambda)$$

- Fill in columns from left to right, with

$$\alpha(j,t) = \sum_{i=1}^{N} \alpha(i, t-1) \cdot a_{ij} \cdot b_j(o_t)$$

  – Same as Viterbi, but sum instead of max (and no backtrace).

Note: because there's a sum, we can't use the trick that replaces probs with costs. For implementation info, see http://digital.cs.usu.edu/~cyan/CS7960/hmm-tutorial.pdf and http://stackoverflow.com/questions/13391625/underflow-in-forward-algorithm-for-hmms .

Example

- Suppose $O=xzy$. Our initially empty table:

	$o_1=x$	$o_2=z$	$o_3=y$
q^1			
q^2			

Filling the first column

	o_1=x	o_2=z	o_3=y
q^1	.6		
q^2	0		

$$\alpha(1,1) = a_{<s>1} \cdot b_1(x) = (1)(.6)$$

$$\alpha(2,1) = a_{<s>2} \cdot b_2(x) = (0)(.1)$$

Starting the second column

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	
q^2	0		

$$\alpha(1,2) = \sum_{i=1}^{N} \alpha(i,1) \cdot a_{i1} \cdot b_{1(z)}$$

$$= \alpha(1,1) \cdot a_{11} \cdot b_1(z) + \alpha(2,1) \cdot a_{21} \cdot b_1(z)$$

$$= (.6)(.7)(.3) + (0)(.5)(.3)$$

$$= .126$$

Finishing the second column

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	
q^2	0	.036	

$$\alpha(2,2) = \sum_{i=1}^{N} \alpha(i,1) \cdot a_{i2} \cdot b_{2(z)}$$

$$= \alpha(1,1) \cdot a_{12} \cdot b_2(z) + \alpha(2,1) \cdot a_{22} \cdot b_2(z)$$

$$= (.6)(.3)(.2) + (0)(.5)(.2)$$

$$= .036$$

Third column and finish

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	.01062
q^2	0	.036	.03906

- Add up all probabilities in last column to get the probability of the entire sequence:

$$P(O|\lambda) = \sum_{i=1}^{N} \alpha(i,T)$$

Learning

- Given *only* the output sequence, learn the best set of parameters $\lambda = (\mathrm{A}, \mathrm{B})$.
- Assume ‘best’ = maximum-likelihood.
- Other definitions are possible, won’t discuss here.

Unsupervised learning

- Training an HMM from an annotated corpus is simple.
  - **Supervised** learning: we have examples labelled with the right ‘answers’ (here, tags): no hidden variables in training.
- Training from unannotated corpus is trickier.
  - **Unsupervised** learning: we have no examples labelled with the right ‘answers’: all we see are outputs, state sequence is hidden.

Circularity

- If we know the state sequence, we can find the best λ.
  - E.g., use MLE: $P(q^j|qi) = \frac{C(qi \rightarrow qj)}{C(qi)}$
- If we know λ, we can find the best state sequence.
  - use Viterbi
- But we don't know either!

Expectation-maximization (EM)

As in spelling correction, we can use EM to bootstrap, iteratively updating the parameters and hidden variables.

- Initialize parameters $\lambda^{(0)}$
- At each iteration k,
  - E-step: Compute **expected counts** using $\lambda^{(k-1)}$
  - M-step: Set $\lambda^{(k)}$ using MLE on the expected counts
- Repeat until λ doesn't change (or other stopping criterion).

Expected counts??

Counting transitions from $q^i \rightarrow q^j$:

- Real counts:
  - count 1 each time we see $q^i \rightarrow q^j$ in true tag sequence.
- Expected counts:
  - With current λ, compute probs of all possible tag sequences.
  - If sequence Q has probability p, count p for each $q^i \rightarrow q^j$ in Q.
  - Add up these fractional counts across all possible sequences.

Example

- Notionally, we compute expected counts as follows:

Possible sequence				Probability of sequence
$Q_1 =$	q^1	q^1	q^1	p_1
$Q_2 =$	q^1	q^2	q^1	p_2
$Q_3 =$	q^1	q^1	q^2	p_3
$Q_4 =$	q^1	q^2	q^2	p_4
Observs:	x	z	y	

Example

- Notionally, we compute expected counts as follows:

Possible sequence				Probability of sequence
$Q_1=$	q^1	q^1	q^1	p_1
$Q_2=$	q^1	q^2	q^1	p_2
$Q_3=$	q^1	q^1	q^2	p_3
$Q_4=$	q^1	q^2	q^2	p_4
Observs:	x	z	y	

$$\hat{C}(q^1 \to q^1) = 2p_1 + p_3$$

Forward-Backward algorithm

- As usual, avoid enumerating all possible sequences.
- **Forward-Backward** (Baum-Welch) algorithm computes expected counts using forward probabilities and **backward probabilities**:

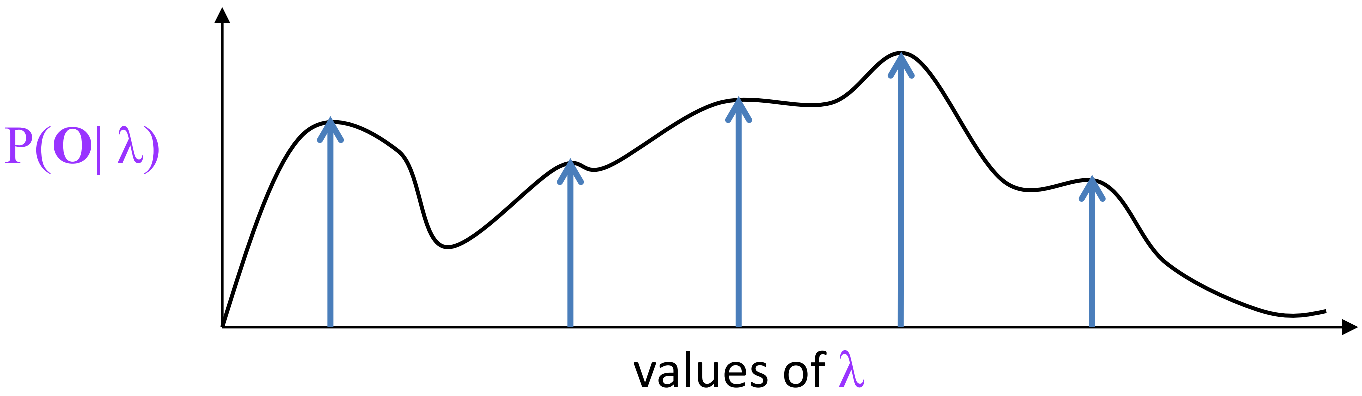
$$\beta(j,t) = P(qt = j, o_{t+1}, o_{t+2}, \ldots oT|\lambda)$$

  - Details, see J&M 6.5

- EM idea is much more general: can use for many latent variable models.

Guarantees

- EM is guaranteed to find a **local** maximum of the likelihood.



- Not guaranteed to find **global** maximum.
- Practical issues: initialization, random restarts, early stopping. Fact is, it doesn't work well for learning POS taggers!