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Invertible Bloom Lookup Tables

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Abstract— **We present a version of the Bloom filter data structure that supports not only the insertion, deletion, and lookup of key-value pairs, but also allows a complete listing of the pairs it contains with high probability, as long the number of key-value pairs is below a designed threshold. Our structure allows the number of key-value pairs to greatly exceed this threshold during normal operation. Exceeding the threshold simply temporarily prevents content listing and reduces the probability of a successful lookup. If entries are later deleted to return the structure below the threshold, everything again functions appropriately. We also show that simple variations of our structure are robust to certain standard errors, such as the deletion of a key without a corresponding insertion or the insertion of two distinct values for a key. The properties of our structure make it suitable for several applications, including database and networking applications that we highlight.**

I. Introduction

The Bloom filter data structure [1] is a well-known way of probabilistically supporting dynamic set membership queries that has been used in a multitude of applications (e.g., see [3]). The key feature of a standard Bloom filter is the way it trades off query accuracy for space efficiency, by using a binary array T (initially all zeroes) and k random hash functions, $h_1, \ldots, h_k$, to represent a set S by assigning $T[h_i(x)] = 1$ for each $x \in S$. To check if $x \in S$ one can check that $T[h_i(x)] = 1$ for $1 \leq i \leq k$, with some chance of a false positive. This representation of S does not allow one to list out the contents of S given only T. In many domains one would benefit from a similar set representation that would also allow listing out the set's contents [9].

In this paper, we are interested not in simply representing a set, but instead in methods for probabilistically representing a lookup table (that is, an associative memory) of key-value pairs, where the keys and values can be represented as fixed-length integers. Unlike previous approaches (e.g., see [2], [4]), we specifically desire a data structure that supports the listing out of all of its key-value pairs. We refer to such a structure as an *invertible Bloom lookup table* (IBLT).

A. Related Work

Our work can be seen as an extension of the invertible Bloom filter data structure of Eppstein and Goodrich [9], modified to store key-value pairs instead of only keys. Our analysis, however, supersedes the analysis of the previous paper in several respects, in terms of efficiency and tightness of the analysis (as well as correcting some small deficiencies). In particular, our analysis demonstrates the natural connection between these data structures and cores of random hypergraphs, similar to the connection found previously for cuckoo hashing and erasure-correcting codes (e.g., see [8], [14]). This provides both a significant constant factor reduction in the required space for the data structure, as well as an important reduction in the error probability (to inverse polynomial, from constant in [9]). In addition, our IBLT supports some usage cases and applications (discussed later in this section) that are not supported by a standard invertible Bloom filter.

While we do not review the large body of work on Bloom filters, two closely related works include Bloomier filters [4] and Approximate Concurrent State Machines (ACSMs) [2], which are structures to store and track key-value pairs. An IBLT has additional features these structures do not have, including listing, graceful handling of data exceeding the listing threshold, and counting multiplicities, which make it useful for several applications where these other structures are insufficient.

Another similar structure is the recently developed counter braid architecture [13], which keeps an updatable count field for a set of flows in a compressed form based on hashing that allows reconstruction of the count for each flow. Unlike an IBLT, however, the flow list must be kept explicitly to read out the flow counts, the lists do not allow for direct lookups of individual values, and their decoding algorithm utilizes a more complex belief propagation. Additional work in the area of approximate counting of a similar flavor but with very different goals from the IBLT includes the well-known CM-sketch [7] and recent work by Price [16].

In parallel with this work, [10] used IBLTs, including IBLTs with negative entries[1], in the context of efficient set reconciliation, further validating the value of this type of structure.

B. Our Results

We present a deceptively simple variation of the Bloom filter data structure that is designed for key-value pairs and further avoids the limitation of previous structures that do not allow the listing of contents. As mentioned above, we call our structure an *invertible Bloom lookup table*, or IBLT for short. Our IBLT supports insertions, deletions, and lookups in $O(k)$ time, where k is the number of random hash functions used (which will typically be a constant in practice). Just as Bloom filters have false positives, our lookup operation works only

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[1]We note the conference paper [10] incorrectly states that that negative entries were not considered in the arxiv'ed version of this paper.

with constant probability, although this probability can be made quite close to 1.

Our data structure also allows for a complete listing of the contained key-value pairs with high probability, whenever the current number n of such pairs lies below a certain threshold capacity t, a parameter that is part of the structure's design. This listing takes $O(t)$ time. One can also use listing as a backup when a standard lookup fails, by performing a listing until finding a value for the desired key.

Our IBLT construction is also space-efficient, requiring space at most linear in t, the threshold number of keys, even if the number, n, of stored key-value pairs grows well beyond t (for example, to polynomial in t) at points in time. One could of course instead keep an actual list of key-value pairs with linear space, but this would require space linear in n, i.e., the *maximum* number of keys, not the target number, t, of keys. Keeping a list also necessarily requires more computationally expensive lookup operations than our approach supports.

We further show that with some additional checksums we can tolerate various natural errors in the system. For example, we can cope with key-value pairs being deleted without first being inserted, or keys being inserted with the same value multiple times, or keys mistakenly being inserted with multiple values simultaneously. Interestingly, together with its contents-listing ability, this error tolerance leads to a number of applications of the IBLT, which we discuss next.

C. Applications and Usage Cases

There are a number of possible applications and usage cases for invertible Bloom lookup tables.

a) Database Reconciliation: Suppose Alice and Bob hold distinct, but similar, copies D_A and D_B of an indexed database D, and they would like to reconcile the differences between D_A and D_B. For example, Alice could hold a current version of D and Bob could hold a backup, or Alice and Bob could represent two different copies of someone's calendar database (say, on a desktop computer and a smartphone) that now need to be symmetrically synchronized.

To achieve such a reconciliation with low overhead, Alice constructs an IBLT, $\mathcal{B}$, for D_A, using indices as keys and checksums of her records as values. She then sends the IBLT $\mathcal{B}$ to Bob, who then deletes index-checksum pairs from $\mathcal{B}$ corresponding to all of his entries in D_B. The remaining key-value pairs corresponding to insertions without deletions identify records that Alice has that Bob doesn't have, and the remaining key-value pairs corresponding to deletions without insertions identify records that Bob has that Alice doesn't have. In addition, as we show, Bob can also use $\mathcal{B}$ to identify records that they both possess but with different checksums. In this way, Alice needs only to send a message $\mathcal{B}$ of size $O(t)$, where t here is an upper bound on the number of differences between D_A and D_B, for Bob to determine the identities of their differences (and a symmetric property holds for a similar message from Bob to Alice).

b) Tracking Network Acknowledgments: Consider a router $\mathcal{R}$ that would like to track TCP sessions passing through $\mathcal{R}$. Each session corresponds to a key, and may have an associated value, such as the source or the destination. When such flows are initiated or terminated in TCP, control messages are passed that can be easily detected, allowing the router to add or remove the flow to the structure. The IBLT supports fast insertions and deletions and can be used to list out the current flows in the system, as long as the number of flows is less than some preset threshold, t. This work can also be offloaded simply by sending a copy of the IBLT to an offline agent if desired. Furthermore, the IBLT can return the value associated with a flow when queried, with constant probability close to 1. Finally, if at any point, the number of flows spikes to well above t, once the total load returns to t or below. the IBLT can again list the flows and perform lookups with the appropriate probabilities Again, this is the key feature of the IBLT; all key-value pairs can be reconstructed with high probability whenever the number of keys is below the design threshold, even if this threshold is *temporarily* exceeded.

In this networking setting, sometimes flows do not terminate properly, leaving them in the data structure when they should disappear. Similarly, initialization messages may not be properly handled, leading to a deletion without a corresponding insertion. We show that the IBLT can be modified to handle such errors with minimal loss in performance. Specifically, we can handle keys that are deleted without being inserted, or keys that erroneously obtain multiple values. Even with such errors, we provide conditions for which all valid flows can still all be listed with high probability. Our experimental results also highlight robustness to these types of errors. (Eventually, of course, such problematic keys should be removed from the data structure; see [2] for some possibilities based on timing structures.)

c) Oblivious Selection from a Table: As a final motivating application, consider a scenario where Alice has outsourced storage of an important indexed table, $\mathcal{T}$, of size n, to a cloud storage server, Bob, because Alice has very limited storage capacity (e.g., Alice may only have a smartphone). Moreover, because her data is sensitive and she knows Bob is honest-but-curious regarding her data, she encrypts each record of T using a secret key, and random nonces, so that Bob cannot determine the contents of any record from its encryption alone. Such encryptions are not sufficient, however, to fully protect the privacy of Alice's data, as recent attacks show that the way Alice accesses her data can reveal its contents (e.g., see [5]). Alice needs a way of hiding any patterns in the way she accesses her data.

Suppose now that Alice would like to do a simple SELECT query on $\mathcal{T}$ and she is confident that the result will have a size at most t, which is much less than n but still more than she can store locally. Thus, she cannot use techniques from private information retrieval [6], as that would either require storing results back with Bob in a way that could reveal selected indices or using yet another server besides Bob. She could use techniques from recent oblivious RAM simulations [11] to obfuscate her access patterns, but doing so would require $O(n \log^2 n)$ I/Os. Therefore, using existing

techniques would be inefficient.

By using an IBLT, she can perform her SELECT query much more efficiently. The advantage comes from the fact that an insertion in an IBLT accesses a random set of cells (that is, memory locations) whose addresses depend (via random hash functions) only on the key of the item being inserted. Alice thus uses all the indices for $\mathcal{T}$ as keys, one for each record, and accesses memory as though inserting each record into an IBLT of size $O(t)$. In fact, Alice only inserts those records that satisfy her SELECT query. However, since Alice encrypts each write using a secret key and random nonces, Bob cannot tell when Alice's write operations are actually changing the records stored in the IBLT, and when a write operation is simply rewriting the same contents of a cell over again re-encrypted with a different nonce. In this way Alice can obliviously create an IBLT of size $O(t)$ that contains the result of her query and is stored by Bob. Then, using existing methods for oblivious RAM simulation [11], she can subsequently obliviously extract the elements from her IBLT using $O(t \log^2 t)$ I/Os. With this approach Bob learns nothing about her data from her access pattern. In addition, the total number of I/Os for her to perform her query is $O(n + t \log^2 t)$, which is linear (and optimal) for any t that is $O(n/\log^2 n)$. We are not currently aware of any other way that Alice can achieve such a result using a structure other than an IBLT.

II. A Simple Version

In this section, we describe and analyze a simple version of the IBLT. In the sections that follow we describe how to augment and extend this simple structure to achieve various additional performance goals.

The IBLT data structure, $\mathcal{B}$, is a randomized data structure storing a set of key-value pairs. It is designed with respect to a threshold number of keys, t; when we say the structure is successful for an operation with high probability it is under the assumption that the actual number of keys in the structure at that time, which we henceforth denote by n, is less than or equal to t. Note that n can exceed t during the course of normal operation, however.

We assume throughout that, as in the standard RAM model, keys and values respectively fit in a single word of memory (which, in practice, could actually be any fixed number of memory words) and that each such word can alternatively be viewed as an integer, character string, floating-point number, etc. Thus, without loss of generality, we view keys and values as positive integers. Space is measured by the number of memory words used.

In many cases we take sums of keys and/or values; we must also consider whether word-value overflow when trying to store these sums in a memory word. (That is, the sum is larger than what fits in a data word.) Such considerations have minimal effects. In most situations, with suitably sized memory words, overflow may never be a consideration. Alternatively, if we work in a system that supports graceful overflows, so that $(x + y) - y = x$ even if the first sum results in an overflow, our approach works with negligible changes. Finally, we can also work modulo some large prime (so that values fit within a memory word) to enforce graceful overflow. These variations have negligible effects on the analysis. However, we point out that in many settings (except in the case where we may have duplicate copies of the same key-value pair), we can use XORs in place of sums in our algorithms, and avoid overflow issues entirely.

A. Operations Supported

Our structure supports the following operations:

- INSERT(x, y): insert the key-value pair, (x, y), into $\mathcal{B}$. This operation always succeeds, assuming that all keys are distinct.
- DELETE(x, y): delete the key-value pair, (x, y), from $\mathcal{B}$. This operation always succeeds, provided $(x, y) \in \mathcal{B}$, which we assume for the rest of this section.
- GET(x): return the value y such that there is a key-value pair, (x, y), in $\mathcal{B}$. If $y =$ null is returned, then $(x, y) \notin \mathcal{B}$ for any value of y. With low (but constant) probability, this operation may fail, returning a "not found" error condition. In this case there may or may not be a key-value pair (x, y) in $\mathcal{B}$.
- LISTENTRIES$()$: list all the key-value pairs being stored in $\mathcal{B}$. With low (inverse polynomial in t) probability, this operation may return a partial list along with an "list-incomplete" error condition.

When an IBLT $\mathcal{B}$ is first created, it initializes a lookup table T of m cells. Each of the cells in T stores a constant number of fields, each of which corresponds to a single memory word. We emphasize that at times the number of key-value pairs in $\mathcal{B}$ can be much larger than m, but the space used for $\mathcal{B}$ remains $O(m)$ words. The INSERT and DELETE methods never fail, whereas the GET and LISTENTRIES methods only guarantee good probabilistic success when $n \leq t$. For our structures we shall generally have $m = O(t)$, and often we can give quite tight analyses on the constant factors required.

B. Data Structure Architecture

Like a standard Bloom filter, an IBLT uses a set of k random[2] hash functions, h_1, h_2, ..., h_k, to determine where key-value pairs are stored. In our case, each key-value pair, (x, y), is placed into cells $T[h_1(x)]$, $T[h_2(x)]$, ... $T[h_t(x)]$. In what follows, for technical reasons[3], we assume that the hashes yield distinct locations. This can be accomplished in various ways, with one standard approach being to split the m cells into k subtables each of size m/k, and having each hash function choose one cell (uniformly) from each subtable. Such splitting does not affect the asymptotic behavior in our analysis.

Each cell contains three fields:

- a count field, which counts the number of entries that have been mapped to this cell,

[2]We assume, for the sake of simplicity in our analysis, that the hash functions are fully random; see the full paper for further discussion.

[3]Incidentally, this same technicality can be used to correct a small deficiency in the paper of Eppstein and Goodrich [9].

- a keySum field, which is the sum of all the keys that have been mapped to this cell,
- a valueSum field, which is the sum of all the values that have been mapped to this cell.

Given these fields, which are initially 0, performing the update operations is fairly straightforward:

- INSERT(x, y):

  for each (distinct) $h_i(x)$, for $i = 1, \ldots, k$ **do**
    add 1 to $T[h_i(x)]$.count
    add x to $T[h_i(x)]$.keySum
    add y to $T[h_i(x)]$.valueSum
  end for

- DELETE(x, y):

  for each (distinct) $h_i(x)$, for $i = 1, \ldots, k$ **do**
    subtract 1 from $T[h_i(x)]$.count
    subtract x from $T[h_i(x)]$.keySum
    subtract y from $T[h_i(x)]$.valueSum
  end for

C. Data Lookups

We perform the GET operation in a manner similar to how membership queries are done in a standard Bloom filter. The details are as follows:

- GET(x):

  for each (distinct) $h_i(x)$, for $i = 1, \ldots, k$ **do**
    if $T[h_i(x)]$.count $= 0$ **then**
      return null
    else if $T[h_i(x)]$.count $= 1$ **then**
      if $T[h_i(x)]$.keySum $= x$ **then**
        return $T[h_i(x)]$.valueSum
      else
        return null
      end if
    end if
  end for
  return "not found"

Recall that for now we assume that all insertions and deletions are done correctly, that is, no insert will be done for an existing key in $\mathcal{B}$ and no delete will be performed for a key-value pair not already in $\mathcal{B}$. With this assumption, if the above operation returns a value y or the null value, then this is the correct response. This method may fail, returning "not found," if it can find no cell that x maps to that holds only one entry. Also, as a value is returned only if the count is 1, overflow of the sum fields is not a concern.

For a key x in $\mathcal{B}$, consider the probability p_0 that each of its hash locations contains no other item. Using the standard analysis for Bloom filters (e.g., see [3]), we find p_0 is:

$$p_0 = \left(1 - \frac{k}{m}\right)^{(n-1)} \approx e^{-kn/m}.$$

One nice interpretation of this is that the number of keys that hash to the cell is approximately a Poisson random variable with mean kn/m, and $e^{-kn/m}$ is the corresponding probability a cell is empty. The probability that a GET for a key that is in $\mathcal{B}$ returns "not found" is therefore approximately

$$(1 - p_0)^k \approx \left(1 - e^{-kn/m}\right)^k,$$

which corresponds to the false-positive rate for a standard Bloom filter. As is standard for these arguments, these approximations can be readily replaced by tight concentration results [3].

The probability that a GET for a key that is *not* in $\mathcal{B}$ returns "not found" instead of null can be found similarly. Here, however, note that every cell hashed to by that key must be hashed to by at least two other keys from $\mathcal{B}$; an empty cell returns a null value, and a cell with just one key hashed returns the true key value, leading to a null return value for a key not in $\mathcal{B}$. Using the same Poisson approximation, we find this probability is

$$\left(1 - e^{-kn/m} - \frac{kn}{m} e^{-kn/m}\right)^k.$$

D. Listing Set Entries

Let us next consider the method for listing the contents of $\mathcal{B}$. We describe this method in a destructive fashion—if one wants a non-destructive method, then one should first create a copy of $\mathcal{B}$ as a backup.

- LISTENTRIES():

  while there's an $i \in [1, m]$ with $T[i]$.count $= 1$ **do**
    add the pair ($T[i]$.keySum , $T[i]$.valueSum) to the output list
    call DELETE($T[i]$.keySum , $T[i]$.valueSum)
  end while

It is a fairly straightforward exercise to implement this method in $O(m)$ time, say, by using a link-list-based priority queue of cells in T indexed by their count fields and modifying the DELETE method to update this queue each time it deletes an entry from $\mathcal{B}$.

If at the end of the while-loop all the entries in T are empty, then we say that the method *succeeded* and we can confirm that the output list is the entire set of entries in $\mathcal{B}$. If, on the other hand, there are some cells in T with non-zero counts, then the method only outputs a partial list of the key-value pairs in $\mathcal{B}$.

This process should appear entirely familiar to those who work with random graphs and hypergraphs. It is exactly the same procedure used to find the 2-core of a random hypergraph (e.g., see [8], [15]). To make the connection, think of the cells as being vertices in the hypergraph, and the key-value pairs as being hyperedges, with the vertices for an edge corresponding to the hash locations for the key. The 2-core is the largest sub-hypergraph that has minimum degree at least 2. The standard "peeling process" finds the 2-core: while there exists a vertex with degree 1, delete it and the corresponding hyperedge. The equivalence between the peeling process and the scheme for LISTENTRIES is immediate. We note that this peeling process is similarly used for various erasure-correcting codes, such as Tornado

codes and its derivatives (e.g., see [14]), that have, in some ways, the same flavor as this construction[4].

Assuming that the cells associated with a key are chosen uniformly at random, we use known results on 2-cores of random hypergraphs. In particular, tight thresholds are known; when the number of hash values k of each is at least 2, there are constants $c_k > 1$ such that if $m > (c_k + \epsilon)n$ for any constant $\epsilon > 0$, LISTENTRIES succeeds with high probability, that is with probability $1-o(1)$. Similarly, if $m < (c_k - \epsilon)n$ for any constant $\epsilon > 0$, LISTENTRIES succeeds with probability $o(1)$. Hence $t = m/c_k$ is (approximately) the design threshold for the IBLT. As can be found in for example [8], [15], these values are given by

$$c_k^{-1} = \sup\left\{\alpha : 0 < \alpha < 1; \forall x \in (0,1), 1 - e^{-k\alpha x^{k-1}} < x\right\}.$$

Table I gives numerical values for these thresholds for $3 \leq k \leq 7$. Here we are not truly concerned with the exact values c_k; it is enough that only linear space is required. It is worth noting that because c_k is relatively small, in practice the choice of the size of the IBLT will generally be determined by the desired probability for a successful GET operation, not the need for listing.

When we design our IBLT, depending on the application, we may want a target probability for succeeding in listing entries. Specifically, we may desire failure to occur with probability $O(t^{-c})$ for a chosen constant c (whenever $n \leq t$). By choosing k sufficiently large and m above the 2-core threshold, we can ensure this; indeed, standard results give that the bottleneck is the possibility of having two edges with the same collection of vertices, giving a failure probability of $O(t^{-k+2})$. The following theorem follows readily from previous work.

Theorem 1: *As long as m is chosen so that $m > (c_k+\epsilon)t$ for some $\epsilon > 0$,* LISTENTRIES *fails with probability $O(t^{-k+2})$ whenever $n \leq t$.*

Finally, up to this point, we have not been concerned with minimizing space for our IBLT structure, noting that it can be done in linear space. In the full paper we discuss various space-saving techniques, including using compressed arrays, quotienting, and irregular graphs.

III. ADDING FAULT TOLERANCE

For cases where there can be deletions for key-value pairs that are not already in $\mathcal{B}$, or values can be inserted for keys that are already in $\mathcal{B}$, we require some fault tolerance. We can utilize a standard approach of adding random checksums to get better fault tolerance.

d) Extraneous Deletions: Let us first consider a case with extraneous deletions only. Specifically, we assume a key-value pair might be deleted without a corresponding insertion; however, here we still assume each key is associated with a single value, and duplicate key-value pairs are not allowed in the system. Such deletions cause a variety of problems for both the GET and LISTENTRIES routines. For example, it is possible for a cell to have an associated count of 1 even if more than one key has hashed to it, so we must re-evaluate our LISTENTRIES routine.

[4]Following this analogy, one could for example, consider *irregular* versions of the IBLT, where different keys utilize a different number of hash values; such a variation could use less space while allowing LISTENTRIES to succeed, or could be used to allow some keys with more hash locations to obtain a better likelihood of a successful lookup. These variations are straightforward and we do not consider the details further here.

k	3	4	5	6	7
c_k	1.222	1.295	1.425	1.570	1.721

TABLE I
THRESHOLDS FOR THE 2-CORE ROUNDED TO FOUR DECIMAL PLACES.

To help deal with these issues, we add to our IBLT structure. We assume that each key x has an additional hash value given by a hash function $G_1(x)$, which in general we assume will take on uniform random values in a range $[1, R]$. We then require each cell has the following additional field:

- a hashkeySum field, which is the sum of the hash values, $G_1(x)$, for all the keys that have been mapped to this cell.

The hashkeySum field must be of sufficiently many bits and the hash function must be sufficiently random to make collisions sufficiently unlikely; this is not hard to achieve in practice. Our insertion and deletion operations must now change accordingly, in that we now must add $G_1(x)$ to each $T[h_i(x)]$.hashkeySum on an insertion and subtract $G_1(x)$ during a deletion. The pseudocode for these and the other operations is given in the full paper.

The hashkeySum field can serve as an extra check. For example, to check when a cell has a count of 1 that it corresponds to a cell without extraneous deletions, we check $G_1(x)$ field against the hashkeySum field. For an error to occur, we must have that a deletion has caused a count of 1 where the count should be higher, and the hashed key values must align so that their sum causes a false check. This probability is clearly at most $1/R$ (using the standard principle of deferred decisions, the "last hash" must take on the precise wrong value for a false check). We will generally assume that R is chosen large enough that we can assume a false match does not occur throughout the lifetime of the data structure, noting that only $O(\log n)$ bits are needed to handle lifetimes that are polynomial in n.

Let us now consider GET operations. The natural approach is to assume that the hashkeySum field will not lead to a false check, as above. In this case, on a GET of a key x, if the count field is 0, and the keySum and hashkeySum are also 0, one should assume that the cell is in fact empty, and return null. Similarly, if the count field is 1, and the keySum and hashkeySum match x and $G_1(x)$, respectively, then one should assume the cell has the right key, and return its value. In fact, if the count field is -1, and after negating keySum and hashkeySum the values match x and $G_1(x)$, respectively, one should assume the cell has the right key, except that it has been deleted instead of inserted! We could return the value, or flag it as an extraneous deletion. Note, however, that we can no longer return null if the count field is 1 but the keySum field does not match x; in this case, there could be, for example, an additional key inserted

and an additional key extraneously deleted from that cell, which would cause the field to not match even if x was hashed to that cell. If we let n be the number of keys either inserted or extraneously deleted in the IBLT, then this reduces the probability of returning null for a key not in $\mathcal{B}$ to $\left(1 - e^{-kn/m}\right)^k$. That is, to return null we must have at least one cell with zero key-value pairs from $\mathcal{B}$ hashing to it, which occurs (approximately) with the given probability (using our Poisson approximation).

For the LISTENTRIES operation, we again use the hashkeySum field to check when a cell has a count of 1 that it corresponds to a cell without extraneous deletions. An error in this check will cause the entire listing operation to fail, so the probability of a false check should be made quite low—certainly inverse polynomial in n. Also, we can make progress in recovering keys with cells with a count of -1 as well, if the cell contains only one extraneously deleted key and no inserted keys. That is, if a cell contains a count of -1, we can negate the count, keySum, and hashkeySum fields, check the hash value against the key to prevent a false match, and if that check passes recover the key and remove it (in this case, add it back in) to the other associated cells. Hence, a cell cannot yield a key during the listing process only if more than one key, either inserted or deleted, has hashed to that cell. This is now exactly the same setting as in the original case of no extraneous deletions, and hence (assuming that no false checks occur!) the same analysis applies, with n representing the number of keys either inserted or extraneously deleted.

e) Multiple Values: A more challenging case for fault tolerance occurs when a key can be inserted multiple times with different values, or inserted and deleted with different values. If a key is inserted multiple times with different values, not only can that key not be recovered, but every cell associated with that key has been *poisoned*, in that it will not be useful for listing keys, as it cannot have a count of 1 even as other values are recovered. (A later deletion of a key-value pair could correct this problem.) The same is true if a key is inserted and deleted with different values, and here the problem is potentially even worse: if a single other key hashes to that cell, the count may be 1 and the keySum and hashkeySum fields will be correct even though the valueSum field will not match the other key's value, causing errors.

Correspondingly, we introduce an additional check for the sum of the values at a cell, using a hash function $G_2(y)$ for the values, and adding the following field:

- a hashvalueSum field, which is the sum of the hash values $G_2(y)$ for all the values that have been mapped to this cell.

One can then check that the hash of the keySum and valueSum take on the appropriate values when the count field of a cell is 1 (or -1) in order to see if listing the key-value pair is appropriate.

The question remains whether the poisoned cells will prevent recovery of key values. Here we modify the goal of LISTENTRIES to return all key-value pairs for all valid keys with high probability—that is, all keys with a single associated value at that time. We first claim that if the invalid keys make up a constant fraction of the n keys that this is not possible under our construction with linear space. A constant fraction of the cells would then be poisoned, and with constant probability each valid key would then hash solely to poisoned cells, in which case the key could not be recovered.

However, it is useful to consider these probabilities, as in practical settings these quantities will determine the probability of failure. For example, suppose γn keys are invalid for some constant γ. By our previous analysis, the fraction of cells that are poisoned is concentrated around $\left(1 - e^{-k\gamma n/m}\right)$, and hence the probability that any specific valid key has all of its cells poisoned is $\left(1 - e^{-k\gamma n/m}\right)^k$. (While there are other possible ways a key could not be recovered, for example if two keys have all but one of their cells poisoned and their remaining cell is the same, this gives a good first approximation for reasonable values, as other terms will generally be lower order when these probabilities are small.) For example, in a configuration we use in our experiments below, we choose $k = 5$, $m/n = 8$, and $\gamma = 1/10$; in this case, the probability of a specific valid key being unrecoverable is approximately $8.16 \cdot 10^{-7}$, which may be quite suitable for practice.

One can also consider a more theoretical asymptotic analysis. In the full version, we show the following result:

Theorem 2: *Suppose there are $n^{1-\beta}$ invalid keys. Let $k = \lceil 1/\beta \rceil + 4$. Then if $m > (c_k + \epsilon)n$ for some $\epsilon > 0$,* LISTENTRIES *succeeds with high probability.*

While such asymptotic analysis provides some useful insights, in practice we expect the heuristic analysis based on considering γn invalid keys will prove more useful for parameter setting and predicting performance.

f) Extensions to Duplicates: Interestingly, using the same approach as for extraneous deletions, our IBLT can handle the setting where the *same* key-value pair is inserted multiple times. Essentially, this means the IBLT is robust to duplicates, or can also be used to obtain a count for key-value pairs that are inserted multiple times. We again use the additional hashkeySum and valueSum fields. When the count field is j for the cell, we take the keySum, hashkeySum, and valueSum fields and divide them by j to obtain the proposed key, value, and corresponding hash. (Here, note we cannot use XORs in place of sums in our algorithms.) If the key hash matches, we assume that we have found the right key and return the lookup value or list the key-value pair accordingly, depending on whether a GET or LISTENTRIES operation is being performed. If it is possible to have the same key appear with multiple values, as above, then we must *also* make use of the hashvalueSum fields, dividing it by j and using it to check that the value is correct as well. For the listing operation, the IBLT deletes j copies

of the key-value pair from the other cells.[5] Thus, duplicate key-value pairs can also be handled with minimal changes.

The one potential issue with duplicate key-value pairs is in the case of word-value overflow for the memory locations containing the sum; in case of overflow, it may be that one does not detect that the key hash matches (and similarly for the hashvalueSum fields). In practice this may limit the number of duplicates that can be tolerated; however, for small numbers of duplicates and suitably sized memory fields, overflow will be a provably rare occurrence.

g) An Example Application: We return to our mirror site application. An IBLT $\mathcal{B}$ from Alice can be used by Bob to find filename-checksum (key-value) pairs where his filename has a different checksum than Alice's. After deleting all his key-value pairs, he lists out the contents of $\mathcal{B}$ to find files that he or Alice has that the other does not. The IBLT might not be empty at this point, however, as the listing process might not have been able to complete due to *poisoned* cells, where deletions were done for keys with values different than Alice's values. To discover these, Bob can re-insert each of his key-value pairs, in turn, to find any that may unpoison a cell in $\mathcal{B}$ (where he immediately deletes ones that don't lead to a new unpoisoned cell). If a new unpoisoned cell is found found (using the G_1 hash function as a check), then Bob can then remove a key-value pair with the same key as his but with a different value (that is, with Alice's value). Note Bob may then also be able to possibly perform more listings of keys that might have been previously unrecovered because of the poisoned cells. Repeating this will discover with high probability all the key-value pairs where Alice and Bob differ.

IV. Simulations and Experiments

We have run a number of simulations to test the IBLT structure and our analysis. In these experiments we have not focused on running time; a practical implementation could require significant optimization. Also, we have not concerned ourselves with issues of word-value overflow. Because of this, there is no need to simulate the data structure becoming overloaded and then deleting key-value pairs, as the state after deletions is determined entirely by the key-value pairs in the system. Instead, we focus on the success probability of the listing of keys and, to a lesser extent, on the success probability for a GET operation. Overall, we have found that the IBLT works quite effectively and the performance matches our theoretical analysis. We provide a few example results. In all of the experiments here, we have chosen to use five hash functions.

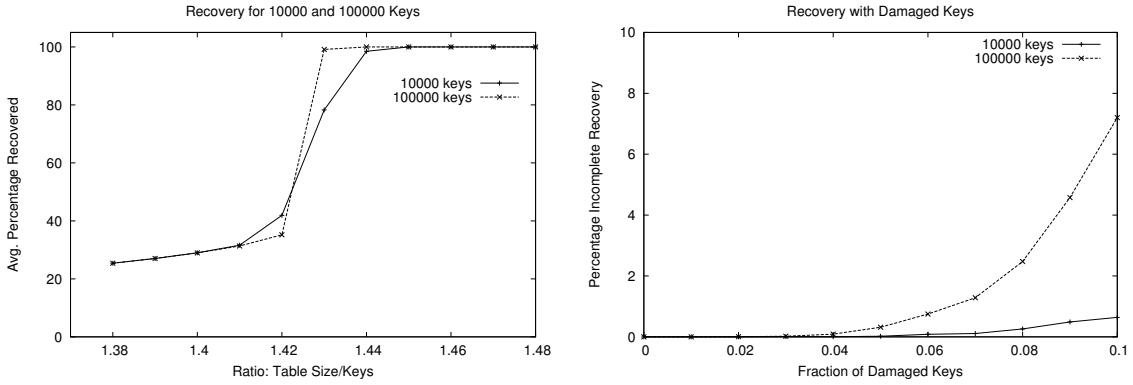
Orur first experiments show that our calculated asymptotic thresholds for decoding from Table I are quite accurate even for reasonably small values. In the setting where there are no duplicate keys or extraneous deletions, we repeatedly performed 20,000 simulations with 10,000 keys, and varied the number of cells. Table I suggests an asymptotic threshold for listing all entries near 14,250. As shown in Figure 1(a), around this point we see a dramatic increase in the average number of key-value pairs recovered when performing our LISTENTRIES operation. At 14,500 cells only two trials failed to recover all key-value pairs, and with 14,600 cells or more all trials successfully recover all key-value pairs. We performed an additional 200,000 trials with 14,600 cells, and again all trials succeeded. Figure 1(a) also shows results from 20,000 simulations with 100,000 keys, where the corresponding threshold should be near 142,500. With more keys, we expect tighter concentration around the threshold, and indeed with 144,000 cells or more all trials successfully recover all key-value pairs. We performed an additional 200,000 trials with with 144,000 cells, and again all trials succeeded.

We acknowledge that more simulations would be required to obtain detailed bounds on the probability of failure to recover all key-value pairs for specific values of the number of key-value pairs and cells. This is equivalent to the well-studied problem of "finite-length analysis" for related families of error-correcting codes. Dynamic programming techniques, as discussed in [12] and subsequent follow-on work, can be applied to obtain such bounds.

Our next tests of the IBLT allow duplicate keys with the same value and extraneous deletions, but without keys with multiple values. Our analysis suggests this should work exactly as with no duplicate or extraneous deletions, and our simulations verify this. In these simulations, we had each key duplicated with probability $1/5$, and each key deleted instead of inserted with probability $1/5$. Using a check on key and value fields, in 20,000 simulations with 10,000 keys and 80,000 cells, a complete listing was obtained every time, and GET operations were successful on average for 97.83 percent of the keys, matching the standard analysis for a Bloom filter. Results were similar with 20,000 runs with 100,000 keys and 800,000 cells, again with complete recovery each time and GET operations successful on average for 97.83 percent of the keys.

Finally, we tested the IBLT with keys that erroneously obtain multiple values. As expected, these keys can prevent recovery of other key-value pairs during listing, but do not impact the success probability of GET operations for other keys. For example, again using a check on key and value fields, in 20,000 simulations with 10,000 keys of which 500 had multiple values and 80,000 cells, the 9500 remaining key-value pairs were recovered 19,996 times; the remaining 4 times all but one of the 9500 key-value pairs was recovered. With 1000 keys with multiple values, the remaining 9000 key-value pairs were recovered 19,872 times, and again the remaining 128 times all but one pair was recovered. The average success rate for GET operations remained 97.83 percent on the valid keys in both cases. We note that with 10,000 keys with 1,000 with multiple values, our previous back-of-the-envelope calculation showed that each valid key would fail with probability roughly $8.16 \cdot 10^{-7}$; hence, with 9,000 other keys, assuming independence we would estimate the probability of complete recovery at approximately 0.9927,

[5]Note that here we are making use of the assumption that the hash locations are distinct for a key; otherwise, the count for the number of copies at this location might not match the number of copies of the key in all the other locations.



(a) (b)

Fig. 1. Plot (a) gives the percentage of key-value pairs recovered around the threshold. Slightly over the theoretical asymptotic threshold, we obtain full recovery of all key-value pairs with LISTENTRIES on all simulations, with greater concentration with more key-value pairs. Plot (b) gives the percentage of trials with incomplete recovery with "damaged" keys that have multiple values. Each data point represents the average of 20,000 simulations.

	Unrecovered Keys			
	0	1	2	3
Experiment 1	99.360	0.640	0.000	0.000
Experiment 2	83.505	14.885	1.520	0.090
Experiment 3	92.800	6.915	0.265	0.020

TABLE II
PERCENTAGE OF TRIALS WITH 1, 2, AND 3 KEYS UNRECOVERED.

closely matching our experimental results. More detailed results are give in Figure 1(b), where we vary the number keys with multiple values for two settings: 10,000 keys and 80,000 cells, and 100,000 keys and 800,000 cells. The results are based on 20,000 trials. As can be seen complete recovery is possible with large numbers of multiple-valued keys in both cases, but the probability of complete recovery becomes worse with larger numbers of keys even if the percentage of invalid keys is the same.

We emphasize that even when complete recovery does not occur in this setting, generally almost all keys with a single value can be recovered. For example, in Table II we consider three experiments. The first is for 10,000 keys, 80,000 cells, and 1,000 keys with duplicate values. The second is the same but with 2,000 keys with duplicate values. The third is for 100,000 keys, 800,000 cells, and 10,000 keys with duplicate values. Over all 20,000 trials for each experiment, in no case were more than 3 valid keys unrecovered. The main point is that with suitable design parameters, even when complete recovery is not possible because of invalid keys, the degradation is minor. We expect this level of robustness will be useful for applications where almost-complete recovery is acceptable.

V. CONCLUSION

We have given an extension to the Bloom filter data structure to key-value pairs and the ability to list out its contents. This structure is deceptively simple, but is able to achieve functionalities and efficiencies that appear to be unique in many respects, based on our analysis derived from recent results on 2-cores in hypergraphs.

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