Approximate Degree and the Complexity of Depth Three Circuits

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Boolean Functions

- Boolean function $f : \{-1, 1\}^n \to \{-1, 1\}$
- $$\mathrm{AND}_n(x) = \begin{cases} -1 \quad \text{(TRUE)} & \text{if } x = (-1)^n \\ 1 \quad \text{(FALSE)} & \text{otherwise} \end{cases}$$

Approximate Degree

- A real polynomial p ϵ-approximates f if

$$|p(x) - f(x)| < \epsilon \quad \forall x \in \{-1, 1\}^n$$

- $\widetilde{\deg}_\epsilon(f) =$ minimum degree needed to ϵ-approximate f
- $\widetilde{\deg}(f) := \widetilde{\deg}_{1/3}(f)$ is the approximate degree of f

Threshold Degree

Definition

Let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function. A polynomial p sign-represents f if $\operatorname{sgn}(p(x)) = f(x)$ for all $x \in \{-1,1\}^n$.

Definition

The threshold degree of f is $\min \deg(p)$, where the minimum is over all sign-representations of f.

- An equivalent definition of threshold degree is $\lim_{\epsilon \nearrow 1} \widetilde{\deg}_\epsilon(f)$.

Why Care About Approximate and Threshold Degree?

Upper bounds on $\widetilde{\deg}_\epsilon(f)$ and $\deg_\pm(f)$ yield **efficient learning algorithms**.

- $\epsilon \approx 1/3$: Agnostic Learning [KKMS05]
- $\epsilon \approx 1 - 2^{-n^\delta}$: Attribute-Efficient Learning [KS04, STT12]
- $\epsilon \to 1$ (i.e., $\deg_\pm(f)$ upper bounds): PAC learning [KS01]

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- Upper bounds on $\widetilde{\deg}_{1/3}(f)$ also imply fast algorithms for **differentially private data release** [TUV12, CTUW14].
- Upper bounds on $\widetilde{\deg}_\epsilon(f)$ and $\deg_\pm(f)$ for small formulas and threshold circuits f yield state of the art **formula size and threshold circuit lower bounds** [Tal17, Forster02].

Why Care About Approximate and Threshold Degree?

Lower bounds on $\widetilde{\deg}_\epsilon(f)$ and $\deg_\pm(f)$ yield lower bounds on:

- **Oracle Separations** [Bei94, BCHTV16]
- **Quantum query complexity** [BBCMW98]
- **Communication complexity** [She08, SZ08, CA08, LS08, She12]
  - Lower bounds hold for a communication problem **related** to f.
  - Via, e.g., a technique called the Pattern Matrix Method [She08].

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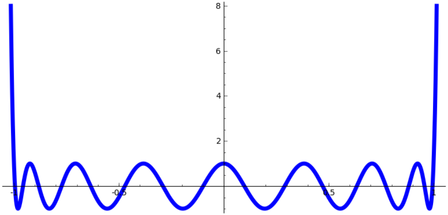
- Lower bounds on $\widetilde{\deg}_\epsilon(f)$ and $\deg_\pm(f)$ also yield efficient **secret-sharing schemes** [BIVW16]

Example 1: The Approximate Degree of AND_n

Example: What is the Approximate Degree of AND_n?

$\widetilde{\deg}(\mathrm{AND}_n) = \Theta(\sqrt{n})$.

- Upper bound: Use **Chebyshev Polynomials**.
- The degree d Chebyshev polynomial T_d satisfies:
  - $|T_d(t)| \leq 1$ for all $t \in [-1, 1]$.
  - $T_d'(\pm 1) = d^2$.



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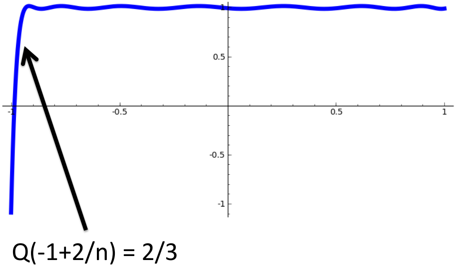
$\widetilde{\deg}(\text{AND}_n) = O(\sqrt{n})$.

- After shifting a scaling, can turn degree $O(\sqrt{n})$ Chebyshev polynomial into a univariate polynomial $Q(t)$ that looks like:

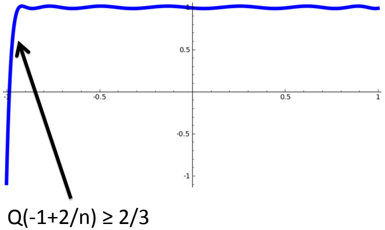


- Define n-variate polynomial p via $p(x) = Q(\sum_{i=1}^{n} x_i/n)$.
- Then $|p(x) - \text{AND}_n(x)| \leq 1/3 \quad \forall x \in \{-1,1\}^n$.

Example: What is the Approximate Degree of AND_n?

[NS92] $\widetilde{\deg}(\mathrm{AND}_n) = \Omega(\sqrt{n})$.

- Lower bound: Use **symmetrization**.
- Suppose $|p(x) - \mathrm{AND}_n(x)| \leq 1/3 \quad \forall x \in \{-1,1\}^n$.
- There is a way to turn p into a univariate polynomial p^{sym} that looks like this:



- Claim 1: $\deg(p^{\text{sym}}) \leq \deg(p)$.
- Claim 2: Markov's inequality $\Longrightarrow \deg(p^{\text{sym}}) = \Omega(n^{1/2})$.

What if ϵ is "somewhat close" to 1?

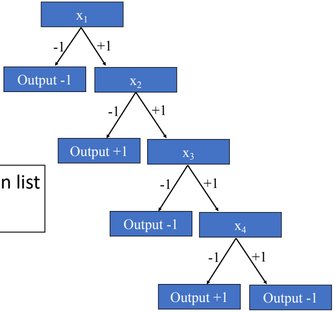
- Fact: $\widetilde{\deg}_{1-1/n}(\mathrm{AND}_n) = 1$.
- Proof: Consider the approximation $1 - 1/n + \sum_{i=1}^{n} x_i/n$.

Example 2: A Function With Large Approximate Degree For ϵ Exponentially Close to 1

Definition

Define the function ODD-MAX-BIT (OMB) via the following procedure: "For $i = 1, \ldots, n$, if $x_i = -1$, halt and output $(-1)^i$."

- OMB is a decision list.
- Any decision list is also a linear-size DNF.



An example decision list on 4 variables.

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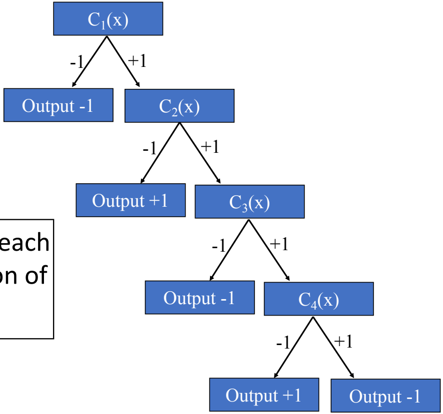
Theorem (Beigel 1992, Klivans and Servedio 2004)

For any $d \geq 0$, $\widetilde{\deg}_\epsilon(\text{OMB}) = d$ for some $\epsilon = 1 - 2^{-\tilde{\Theta}(n/d^2)}$.

- Special cases:
  - $\deg_\pm(\text{OMB}) = \widetilde{\deg}_{1-2^{-\Theta(n)}}(\text{OMB}) = 1$.
  - $\widetilde{\deg}_\epsilon(\text{OMB}) = \tilde{\Theta}(n^{1/3})$ for $\epsilon = 1 - 2^{-n^{1/3}}$.
  - $\widetilde{\deg}_{1/3}(\text{OMB}) = \tilde{\Theta}(n^{1/2})$.

k-Decision Lists



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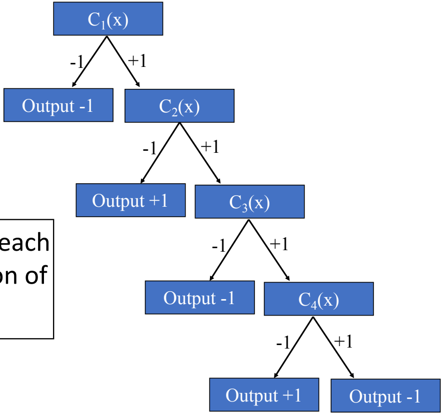

In a k-decision list, each $C_i(x)$ is a conjunction of width k.

- Any k-decision list of length ℓ list is computed by a depth-3 circuit of bottom fan-in $O(k)$ and size $O(\ell)$.

Our Main Result

Theorem (Main Theorem)

For **any** *(large) constant* $\Gamma > 0$ *and (small) constant* $\delta > 0$*, there is an* $O(\log n)$*-decision list* f *of length poly*(n) *satisfying the following:* $\widetilde{\deg}_\epsilon(f) \geq n^{1/2-\delta}$ *for* $\epsilon = 1 - 2^{-n^\Gamma}$*.*

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- Compare to prior work:

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- In Main Theorem, $\deg_\pm(f) = O(\log n)$ and $\widetilde{\deg}(f) = \tilde{\Theta}(n^{1/2})$.
- So our f can be sign-represented by a **very** low degree polynomial, but any polynomial of degree $\ll \widetilde{\deg}(f)$ must incur **extremely** large error (superexponentially close to 1).
- Proving this type of result requires fundamentally new techniques.

Two Main Motivations for Our Main Result

First Motivation: PAC Learning DNFs

- The fastest known algorithm for PAC learning DNFs runs in time $2^{\tilde{O}(n^{1/3})}$ [Klivans and Servedio 2001].
- Follows from the fact that for any DNF f, $\deg_{\pm}(f) = \tilde{O}(n^{1/3})$.

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- How do Klivans and Servedio prove this?
  - First, they turn any DNF into a (generalization of) a k-decision list, for some $k = \tilde{O}(n^{1/3})$.
  - Second, they observe that any k-decision list f satisfies $\deg_{\pm}(f) \leq k$.
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- Klivans and Servedio ask: for any DNF f, is it possible that $\widetilde{\deg}_{\epsilon}(f) \le \tilde{O}(n^{1/3})$ for $\epsilon = 1 - 2^{-n^{1/3}}$?
- An affirmative answer would yield a much simpler DNF learning algorithm.

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- An affirmative answer would yield a much simpler DNF learning algorithm.
- Our Main Theorem comes close to a <u>negative</u> resolution of their question.

Second Motivation: Complexity of AC^0

- $\mathbf{PP}$ is the class of all languages solvable by polynomial time randomized algorithms that output the correct answer with probability strictly better than $1/2$.
- $\mathbf{PP}$ has a natural communication analog, $\mathbf{PP}^{\mathsf{cc}}$.
- Why is $\mathbf{PP}^{\mathsf{cc}}$ important?
- $\mathbf{PP}^{\mathsf{cc}}(F)$ characterizes the margin complexity and discrepancy of F.
- $\mathbf{PP}^{\mathsf{cc}}(F) \geq d \Longrightarrow F$ is not computed by Majority-of-Threshold Circuits of size 2^d.
- Open question: How big can $\mathbf{PP}^{\mathsf{cc}}(F)$ be for an AC^0 function F? Can it be $\Omega(n)$?

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- (Sherstov 2008): If $\widetilde{\deg}_\epsilon(f) \geq d$ for $\epsilon = 1 - 2^{-d}$, then f can be turned into a related function F satisfying $\mathbf{PP}^{\mathsf{cc}}(F) \geq d$.

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- Open question: How big can $\mathbf{PP}^{\mathsf{cc}}(F)$ be for an AC^0 function F?
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  - (Bun and Thaler 2015): A depth-3 circuit F with $\mathbf{PP}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(n^{2/5})$.
  - (Sherstov 2015): A depth-3 circuit F $\mathbf{PP}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(n^{3/7})$ and a depth-4 circuit F with $\mathbf{PP}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(n^{1/2})$.

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  - (Sherstov 2015): A depth-3 circuit F $\mathbf{PP}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(n^{3/7})$ and a depth-4 circuit F with $\mathbf{PP}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(n^{1/2})$.
- **Our work**: For any constant $\delta > 0$, there is a depth-3 circuit F with $\mathbf{PP}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(n^{1/2-\delta})$.

Prior Techniques: Proving Hardness Amplification Theorems For Block-Composed Functions

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These theorems show that $g \circ f$ is "harder to approximate" by low-degree polynomials than is f alone.

Here, $g \circ f = g(f, \ldots, f)$ is the block-composition of g and f.

Hardness-Amplification Theorems From Prior Work

Theorem (Sherstov 2010)

Let f be a Boolean function with $\widetilde{\deg}_{1/2}(f) \geq d$. Let $F = \oplus_t \circ f$, where $\oplus_t$ is the parity function on t bits. Then $\widetilde{\deg}_{1-2^{-t}}(F) \geq d \cdot t$.

Theorem (Bun and Thaler 2014)

Let f be a Boolean function with $\widetilde{odeg}_{-,1/2}(f) \geq d$. Let $F = \mathrm{OR}_t \circ f$. Then $\widetilde{\deg}_{1-2^{-t}}(F) \geq d$.

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Theorem (Thaler 2014)

Let f be a Boolean function with $\widetilde{odeg}_{+,1/2}(f) \geq d$. Let $F = \mathsf{OMB}_t \circ f$. Then $\widetilde{\deg}_{1-2^{-t}}(F) \geq d$.

Our Techniques: Beyond Block-Composed Functions

An $O(\log n)$-Decision List Harder to Approximate than OMB?

Theorem (Beigel94, Thaler14)

Let $F = \mathsf{OMB}_t \circ \mathrm{OR}_b$. *Then* $\widetilde{\deg}_{1-2^{-t}}(F) \geq \sqrt{b}$. *E.g., if* $t = n^{1/3}$ *and* $b = n^{2/3}$, *then* $\deg_\epsilon(F) \geq n^{1/3}$ *for* $\epsilon = 1 - 2^{-n^{1/3}}$.

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- Our goal is to modify $\text{OMB}_t \circ \text{OR}_b$ to obtain a function f that is much harder to approximate by low-degree polynomials, while still ensuring that f is computed by an $O(\log n)$-decision list.

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- First attempt: Letting $\oplus_k$ denote the Parity function on k bits, consider $F := \oplus_k \circ \mathsf{OMB}_t \circ \mathrm{OR}_b$.
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  - This is a k-decision list of length n^k.
- Unfortunately, this is too easy to approximate.
  - Let p approximate $\mathrm{OMB}_b \circ \mathrm{OR}_t$ to error $1 - \epsilon$.
  - Then the polynomial $q(x_1, \ldots, x_k) = \prod_{i=1}^{k} p(x_i)$ approximates $F(x_1, \ldots, x_k)$ to error $1 - \epsilon^k$.

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  - Note: q treats each of the k "blocks" x_i independently, and outputs the products of the k results.

Moving Beyond Block-Composition

- Our F first "pre-processes" its input $(x_1, \ldots, x_k)$ to obtain values $(u_1, \ldots, u_k) \in \{-1, 1\}^{(t \cdot b) \times k}$, which are then fed into $\oplus_k \circ \mathsf{OMB}_t \circ \mathrm{OR}_b$.
- The pre-processing introduces dependencies between blocks.
  - This ensures that an approximating polynomial for F will be unable to treat them independently.
  - But the pre-processing is "mild" enough that F is an $O(\log n)$-decision list of length n^k.
  - The larger k is, the better our lower bound for F (i.e., the lower bound holds for a larger Γ and a smaller δ).

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  - The larger k is, the better our lower bound for F (i.e., the lower bound holds for a larger Γ and a smaller δ).
- Idea for $k = 2$.
  - F takes two input "blocks" (x_1, x_2), with $x_1 \in \{-1,1\}^{t \cdot b}$, and $x_2 \in \{-1,1\}^{t \cdot b \cdot \log_2(t \cdot b)}$.
  - Turn (x_1, x_2) into $(u_1, u_2) \in \{-1,1\}^{t \cdot b} \times \{-1,1\}^{t \cdot b}$ as follows:

Moving Beyond Block-Composition

- Our F first "pre-processes" its input $(x_1, \ldots, x_k)$ to obtain values $(u_1, \ldots, u_k) \in \{-1, 1\}^{(t \cdot b) \times k}$, which are then fed into $\oplus_k \circ \mathsf{OMB}_t \circ \mathrm{OR}_b$.
- The pre-processing introduces dependencies between blocks.
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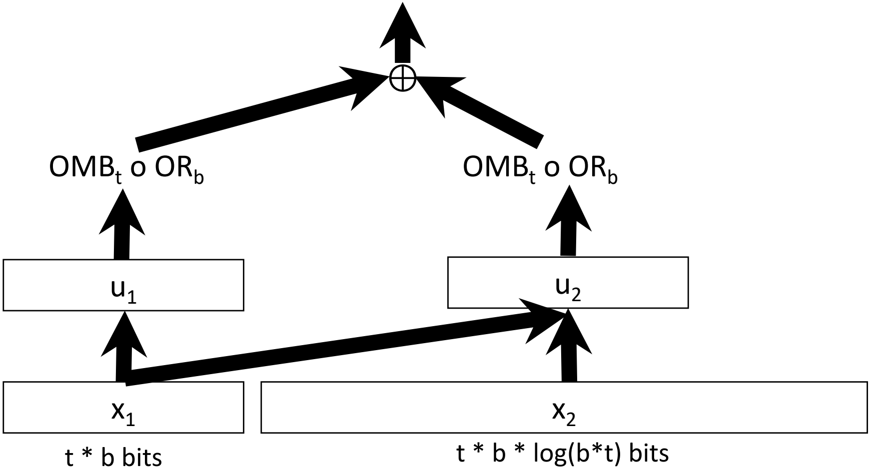
  - $u_1 = x_1$.
  - Let $i^* \in \{1, \ldots, t\}$ be the largest value such that $x_{1, i^*} = -1$.
  - u_2 is obtained from x_2 by testing each consecutive sequence of $\log_2(tb)$ bits for equality with (the binary representation of) i^*.

Schematic of Our Hard-To-Approximate $O(\log n)$-Decision List for $k = 2$



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- (Bun and Thaler, 2017): A different hardness amplification technique that moves beyond block-composed functions.
  - For any constant $\delta > 0$, yielded a nearly-optimal $\Omega(n^{1-\delta})$ lower bound on the approximate degree of AC^0 (specifically, depth $\log(1/\delta)$).
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- Can we extend our lower bound for $O(\log n)$-decision lists to DNFs, answering the question of Klivans and Servedio?

Thank you!