The Polynomial Method Strikes Back: Tight Quantum Query Bounds Via Dual Polynomials

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Boolean Functions

- Boolean function $f : \{-1, 1\}^n \to \{-1, 1\}$
- $$\mathrm{AND}_n(x) = \begin{cases} -1 \quad \text{(TRUE)} & \text{if } x = (-1)^n \\ 1 \quad \text{(FALSE)} & \text{otherwise} \end{cases}$$

Approximate Degree

- A real polynomial p ϵ-approximates f if

$$|p(x) - f(x)| < \epsilon \quad \forall x \in \{-1,1\}^n$$

- $\widetilde{\deg}_\epsilon(f) =$ minimum degree needed to ϵ-approximate f
- $\widetilde{\deg}(f) := \deg_{1/3}(f)$ is the approximate degree of f

Example 1: The Approximate Degree of AND_n

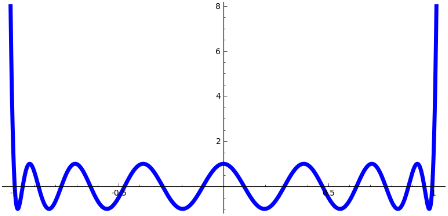
Example: What is the Approximate Degree of AND_n?

$\widetilde{\deg}(\mathrm{AND}_n) = \Theta(\sqrt{n})$.

- Upper bound: Use **Chebyshev Polynomials**.
- Markov's Inequality: Let $G(t)$ be a univariate polynomial s.t. $\deg(G) \leq d$ and $\max_{t \in [-1,1]} |G(t)| \leq 1$. Then

$$\max_{t \in [-1,1]} |G'(t)| \leq d^2.$$

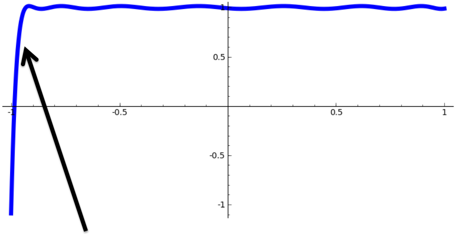
- Chebyshev polynomials are the extremal case.



Example: What is the Approximate Degree of AND_n?

$\widetilde{\deg}(\mathrm{AND}_n) = O(\sqrt{n})$.

- After shifting a scaling, can turn degree $O(\sqrt{n})$ Chebyshev polynomial into a univariate polynomial $Q(t)$ that looks like:

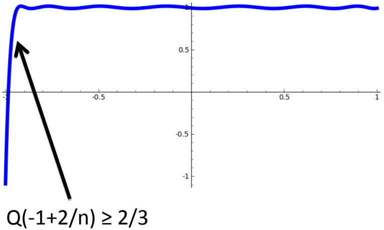


- Define n-variate polynomial p via $p(x) = Q(\sum_{i=1}^{n} x_i/n)$.
- Then $|p(x) - \mathrm{AND}_n(x)| \leq 1/3 \quad \forall x \in \{-1,1\}^n$.

Example: What is the Approximate Degree of AND_n?

[NS92] $\widetilde{\deg}(\mathrm{AND}_n) = \Omega(\sqrt{n})$.

- Lower bound: Use **symmetrization**.
- Suppose $|p(x) - \mathrm{AND}_n(x)| \leq 1/3 \quad \forall x \in \{-1,1\}^n$.
- There is a way to turn p into a univariate polynomial p^{sym} that looks like this:



- Claim 1: $\deg(p^{\mathsf{sym}}) \leq \deg(p)$.
- Claim 2: Markov's inequality $\implies \deg(p^{\mathsf{sym}}) = \Omega(n^{1/2})$.

Why Care about Approximate Degree?

Applications of $\widetilde{\deg}$ Upper Bounds

Upper bounds on $\widetilde{\deg}_\epsilon(f)$ yield efficient learning algorithms.

- $\epsilon \approx 1/3$: Agnostic Learning [KKMS05]
- $\epsilon \approx 1 - 2^{-n^\delta}$: Attribute-Efficient Learning [KS04, STT12]
- $\epsilon \to 1$ (i.e., threshold degree, $\deg_\pm(f)$): PAC learning [KS01]

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- Upper bounds on $\widetilde{\deg}_{1/3}(f)$ also:
  - Imply fast algorithms for differentially private data release [TUV12, CTUW14].

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- Upper bounds on $\widetilde{\deg}_{1/3}(f)$ also:
  - Imply fast algorithms for differentially private data release [TUV12, CTUW14].
  - Underly the best known lower bounds on formula complexity and graph complexity [Tal2014, 2016a, 2016b]

This Talk: Two Focuses Involving $\widetilde{\deg}$ Lower Bounds

- Focus 1: A nearly optimal bound on the approximate degree of AC^0, and its applications [BT17].
- Focus 2: Proving tight quantum query lower bounds for specific functions [BKT17].

First Focus: Approximate Degree of AC^0

- Approximate degree is a key tool for understanding AC^0.
- At the heart of the best known bounds on the complexity of AC^0 under measures such as:
  - Multi-Party (Quantum) Communication Complexity
  - Approximate Rank
  - Sign-rank $\approx$ Unbounded Error Communication (UPP)
  - Discrepancy $\approx$ Margin complexity
  - Majority-of-Threshold circuit size
  - Threshold-of-Majority circuit size
  - and more.

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Problem 1: Is there a function on n variables that is in AC^0, and has approximate degree $\Omega(n)$?

Approximate Degree of AC^0: Details

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  - More precisely, circuit depth is $O(\log(1/\delta))$.

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- Our result: For any constant $\delta > 0$, a function in AC^0 with approximate degree $\Omega(n^{1-\delta})$.
  - More precisely, circuit depth is $O(\log(1/\delta))$.
  - Lower bound also applies to DNFs of polylogarithmic width (and quasipolynomial size).

Applications

- Nearly optimal $\Omega(n^{1-\delta})$ lower bounds on quantum communication complexity of AC^0.
- Essentially optimal (quadratic) separation of certificate complexity and approximate degree.
- Better secret sharing schemes with reconstruction in AC^0.

Second Focus: Quantum Query Complexity

- In the quantum query model, a quantum algorithm is given query access to the bits of an input x.
- Goal: compute some function f of x while minimizing the number of queried bits.
- Most quantum algorithms were discovered in or can easily be described in the query setting.

Connecting $\widetilde{\deg}$ and Quantum Query Complexity

- Let $\mathcal{A}$ be a quantum algorithm making at most T queries.
- [BBC$^+$01] there is a polynomial p of degree $2T$ such that

$$p(x) = \Pr[\mathcal{A}(x) = 1].$$

  - So $\mathcal{A}$ computes f to error $\epsilon \Longrightarrow 2p(x) - 1$ approximates f to error 2ϵ.
  - So $\widetilde{\deg}(f)$ is a lower bound on the quantum query complexity of f.
- This is called the **polynomial method** in quantum query complexity.

Our Results

Problem	Prior Upper Bound	Our Lower Bound	Prior Lower Bound
k-distinctness	$O(n^{3/4-1/(2^{k+2}-4)})$	$\tilde{\Omega}(n^{3/4-1/(2k)})$	$\tilde{\Omega}(n^{2/3})$
Image Size Testing	$O(\sqrt{n}\log n)$	$\tilde{\Omega}(\sqrt{n})$	$\tilde{\Omega}(n^{1/3})$
k-junta Testing	$O(\sqrt{k}\log k)$	$\tilde{\Omega}(\sqrt{k})$	$\tilde{\Omega}(k^{1/3})$
SDU	$O(\sqrt{n})$	$\tilde{\Omega}(\sqrt{n})$	$\tilde{\Omega}(n^{1/3})$
Shannon Entropy	$\tilde{O}(\sqrt{n})$	$\tilde{\Omega}(\sqrt{n})$	$\tilde{\Omega}(n^{1/3})$

Our lower bounds on quantum query complexity and $\widetilde{\deg}$ vs. prior work.

Problem	Prior Upper Bound	Our Upper and Lower Bounds	Prior Lower Bound
Surjectivity	$\tilde{O}(n^{3/4})$	$\tilde{O}(n^{3/4})$ and $\tilde{\Omega}(n^{3/4})$	$\tilde{\Omega}(n^{2/3})$

Our bounds on the approximate degree of Surjectivity vs. prior work.

Lower Bound Methods in Quantum Query Complexity

- Since 2002, the positive-weights adversary method, and the newer negative-weights adversary method have been tools of choice for proving quantum query lower bounds.
  - Negative-weights method can prove a tight lower bound for any function [Rei11, LMR+11].
  - But is often challenging to apply to specific functions.
- Quantum query bounds proved via approximate degree "lift" to communication lower bounds [She11].
  - Not known to hold for adversary methods.

Ruminations on the Polynomial Method

- Intuitively, how do we resolve questions that have resisted adversary methods?
  - A key fact exploited in our analysis is:

Fact (1)

Any polynomial $p\colon \{-1,1\}^n \to \mathbb{R}$ *satisfying the following conditions requires degree* $\Omega(n^{1/4})$*:*

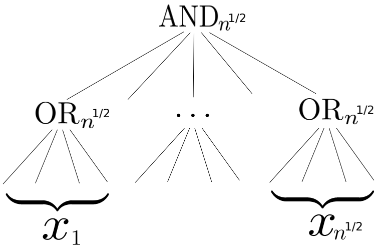
$$\begin{cases} |p(x) - \mathrm{OR}_n(x)| \le 1/3 & \textit{if } |x| \le n^{1/4} \\ |p(x)| \le \exp(|x| \cdot n^{-1/4}) & \textit{if } |x| > n^{1/4}. \end{cases}$$

  - Fact (1) is "non-quantum" because any quantum query algorithm always produces polynomials bounded in $[0,1]$.
- Reasoning about such "non-quantum" polynomials seems difficult to capture by adversary methods.

Prior Work: The Method of Dual Polynomials and the AND-OR Tree

Beyond Symmetrization

- Symmetrization is "lossy": in turning an n-variate poly p into a univariate poly p^{sym}, we throw away information about p.
- **Challenge Problem**: What is $\widetilde{\deg}(\text{AND-OR}_n)$?



History of the AND-OR Tree

Theorem

$\widetilde{\deg}(\text{AND-OR}_n) = \Theta(n^{1/2})$.

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Tight Upper Bound of $O(n^{1/2})$

[HMW03] via quantum algorithms
[BNRdW07] different proof of $O(n^{1/2} \cdot \log n)$ (via error reduction+composition)
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Tight Lower Bound of $\Omega(n^{1/2})$

[BT13] and [She13] via the method of dual polynomials

Linear Programming Formulation of Approximate Degree

What is best error achievable by **any** degree d approximation of f?
Primal LP (Linear in ϵ and coefficients of p):

$$\begin{aligned} \min_{p,\epsilon} \quad & \epsilon \\ \text{s.t.} \quad & |p(x) - f(x)| \leq \epsilon && \text{for all } x \in \{-1,1\}^n \\ & \deg p \leq d \end{aligned}$$

Dual LP:

$$\begin{aligned} \max_{\psi} \quad & \sum_{x \in \{-1,1\}^n} \psi(x) f(x) \\ \text{s.t.} \quad & \sum_{x \in \{-1,1\}^n} |\psi(x)| = 1 \\ & \sum_{x \in \{-1,1\}^n} \psi(x) q(x) = 0 && \text{whenever } \deg q \leq d \end{aligned}$$

Dual Characterization of Approximate Degree

Theorem: $\deg_\epsilon(f) > d$ iff there exists a "dual polynomial" $\psi \colon \{-1,1\}^n \to \mathbb{R}$ with

(1) $$\sum_{x \in \{-1,1\}^n} \psi(x) f(x) > \epsilon$$ "high correlation with f"

(2) $$\sum_{x \in \{-1,1\}^n} |\psi(x)| = 1$$ "L_1-norm 1"

(3) $$\sum_{x \in \{-1,1\}^n} \psi(x) q(x) = 0, \text{ when } \deg q \leq d$$ "pure high degree d"

A **lossless** technique. Strong duality implies any approximate degree lower bound can be witnessed by dual polynomial.

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Example: $2^{-n} \cdot$ PARITY$_n$ witnesses the fact that $\widetilde{\deg}_\epsilon(\text{PARITY}_n) = n$ for any $\epsilon < 1$.

Goal: Construct an explicit dual polynomial $\psi_{\textsf{AND-OR}}$ for AND-OR

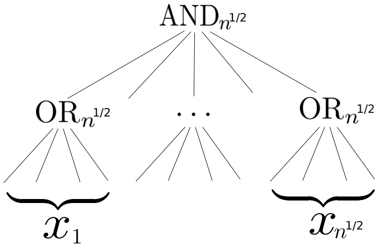
Constructing a Dual Polynomial

- By [NS92], there are dual polynomials
  $\psi_{\textsf{OUT}}$ for $\widetilde{\deg}\,(\mathrm{AND}_{n^{1/2}}) = \Omega(n^{1/4})$ and
  $\psi_{\textsf{IN}}$ for $\widetilde{\deg}\,(\mathrm{OR}_{n^{1/2}}) = \Omega(n^{1/4})$
- Both [She13] and [BT13] combine $\psi_{\textsf{OUT}}$ and $\psi_{\textsf{IN}}$ to obtain a dual polynomial $\psi_{\textsf{AND-OR}}$ for $\mathrm{AND\text{-}OR}$.
- The combining method was proposed in earlier work by [SZ09, Lee09, She09].

The Combining Method [SZ09, She09, Lee09]

$$\psi_{\textsf{AND-OR}}(x_1, \ldots, x_{n^{1/2}}) := C \cdot \psi_{\textsf{OUT}}(\ldots, \mathrm{sgn}(\psi_{\textsf{IN}}(x_i)), \ldots) \prod_{i=1}^{n^{1/2}} |\psi_{\textsf{IN}}(x_i)|$$

(C chosen to ensure $\psi_{\textsf{AND-OR}}$ has L_1-norm 1).



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Must verify:

1. $\psi_{\textbf{AND-OR}}$ has pure high degree $\geq n^{1/4} \cdot n^{1/4} = n^{1/2}$.
2. $\psi_{\textbf{AND-OR}}$ has high correlation with $\mathrm{AND\text{-}OR}$.

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2. $\psi_{\textsf{AND-OR}}$ has high correlation with AND-OR. [BT13, She13]

Recent Progress on the Complexity of AC^0: Applying the Method of Dual Polynomials to Block-Composed Functions

(Negative) One-Sided Approximate Degree

- Negative one-sided approximate degree is an intermediate notion between approximate degree and threshold degree.
- A real polynomial p is a negative one-sided ϵ-approximation for f if

$$|p(x) - 1| < \epsilon \quad \forall x \in f^{-1}(1)$$

$$p(x) \leq -1 \quad \forall x \in f^{-1}(-1)$$

- $\widetilde{\text{odeg}}_{-,\epsilon}(f) =$ min degree of a negative one-sided ϵ-approximation for f.
- Examples: $\widetilde{\text{odeg}}_{-,1/3}(\text{AND}_n) = \Theta(\sqrt{n})$; $\widetilde{\text{odeg}}_{-,1/3}(\text{OR}_n) = 1$.

Recent Theorems

Theorem (BT13, She13)

Let f be a Boolean function with $\widetilde{odeg}_{-,1/2}(f) \geq d$. Let $F = \mathrm{OR}_t(f, \ldots, f)$. Then $\widetilde{\deg}_{1/2}(F) \geq d \cdot \sqrt{t}$.

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Let f be a Boolean function with $\widetilde{odeg}_{-,1/2}(f) \geq d$. *Let* $F = \mathrm{OR}_t(f, \ldots, f)$. *Then* $\deg_{\pm}(F) = \Omega(\min\{d, t\})$.

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Theorem (BCHTV16)

Let f be a Boolean function with $\widetilde{\deg}_{1/2}(f) \geq d$. Let $F = GAPMAJ_t(f, \ldots, f)$. Then $\deg_{\pm}(F) \geq \Omega(\min\{d, t\})$.

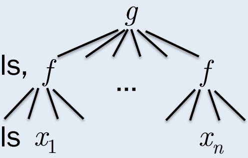
Reminder

Problem 1: Is there a function on n variables that is in AC^0, and has approximate degree $\Omega(n)$?

Our Techniques

Hardness Amplification in AC^0

<u>Theorem Template:</u> If f is "hard" to approximate by low-degree polynomials, then $F = g \circ f$ is "even harder" to approximate by low-degree polynomials



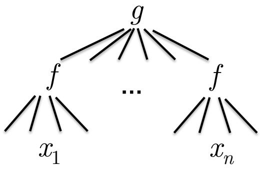
<u>"Block Composition Barrier"</u>

Robust approximations, i.e.,

$$\widetilde{\deg}(g \circ f) \leq O(\widetilde{\deg}(g) \cdot \widetilde{\deg}(f))$$

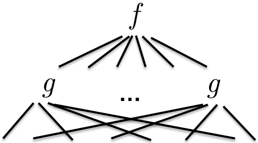
imply that block composition *cannot* increase approximate degree as a function of n

Around the Block-Composition Barrier



Prior work:

- Hardness amplification "from the top"
- Block composed functions



This work:

- Hardness amplification "from the bottom"
- Non-block-composed functions

A General Hardness Amplification Result

Theorem (Strong Hardness Amplification Within AC^0)

Let $f\colon \{-1,1\}^n \to \{-1,1\}$

- *be computed by an AC^0 circuit of depth k, and*
- $\widetilde{\deg}(f) \geq d$.

Then there exists an F on $O(n \log^2 n)$ variables that

- *is computed by an AC^0 circuit of depth $k+3$, and*
- $\widetilde{\deg}(F) \geq n^{1/2} \cdot d^{1/2}$

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Remarks:

- E.g.: If $f = \mathrm{AND}$, then $\widetilde{\deg}(F) \geq n^{3/4}$.
- Recursive application yields $\Omega(n^{1-\delta})$ bound for AC^0 function.
- Analogous result holds for monotone DNF.

Idea of the Hardness Amplification Construction

Idea of the Hardness-Amplifying Construction

- Consider the function SURJECTIVITY: $\{-1,1\}^n \to \{-1,1\}$.
  - Let $n = N \log R$. SURJ interprets its input x as a list of N numbers $(x_1, \ldots, x_N)$ from a range $[R]$.
  - $\text{SURJ}_{R,N}(x) = -1$ if and only if every element of the range $[R]$ appears at least once in the list.
- When we apply Main Theorem to $f = \text{AND}_R$, the "harder" function F is precisely $\text{SURJ}_{R,N}$.
- We show that $\widetilde{\deg}(\text{SURJ}_{R,N}) = \tilde{\Theta}(R^{1/4} \cdot N^{1/2})$.
  - If $R = \Theta(N)$, this is $\tilde{\Theta}(n^{3/4})$.

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  - If $R = \Theta(N)$, this is $\tilde{\Theta}(n^{3/4})$.
- For convenience: let's change the domain and range of all Boolean functions to $\{0,1\}^n$ and $\{0,1\}$.

Resolving the Approximate Degree of SURJ

The $\tilde{O}(R^{1/4} \cdot N^{1/2})$ Upper Bound For SURJ: First Try

- Let's start with how to achieve a (loose) bound of $\widetilde{\deg}(\mathsf{SURJ}_{R,N}) = \tilde{O}(R^{1/2} \cdot N^{1/2})$.

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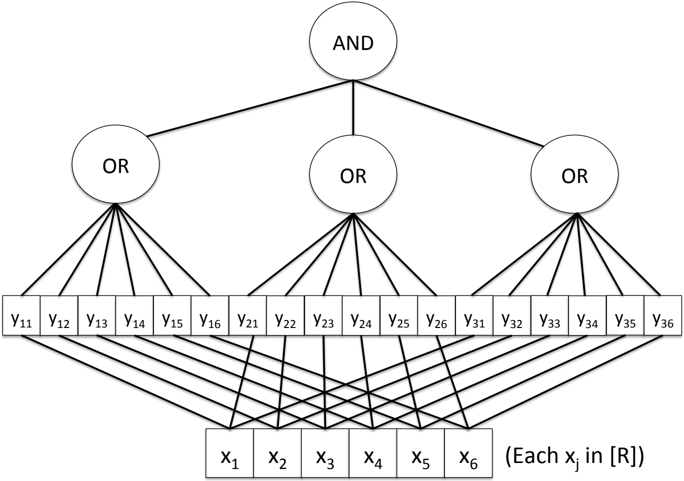
  - Let
$$y_{ij} = \begin{cases} 1 \text{ if } x_j = i \\ 0 \text{ otherwise} \end{cases}$$
  - Then

$$\mathsf{SURJ}(x) = \mathrm{AND}_R(\mathrm{OR}_N(y_{1,1}, \ldots, y_{1,N}), \ldots, \mathrm{OR}_N(y_{R,1} \ldots, y_{R,N})).$$

SURJ Illustrated $(R = 3, N = 6)$



(Each x_j in [R])

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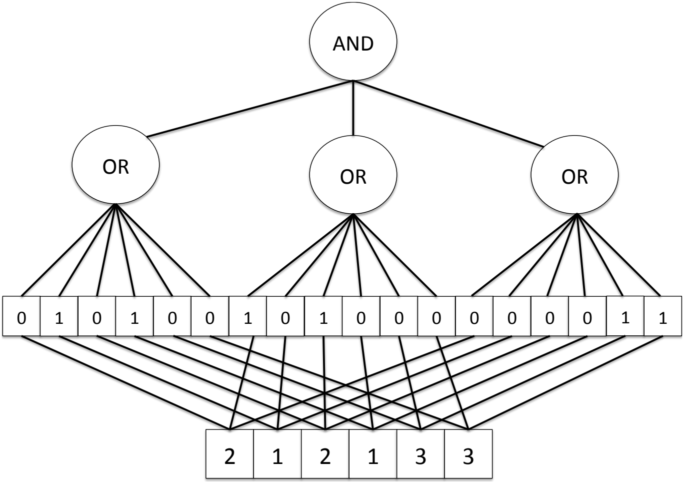

The Upper Bound For SURJ: First Try

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The Upper Bound For SURJ: First Try

- Let's start with how to achieve a (loose) bound of $\widetilde{\deg}(\mathsf{SURJ}_{R,N}) = \tilde{O}(R^{1/2} \cdot N^{1/2})$.
  - Let
$$y_{ij} = \begin{cases} 1 \text{ if } x_j = i \\ 0 \text{ otherwise} \end{cases}$$
  - Then

$$\mathsf{SURJ}(x) = \mathrm{AND}_R(\mathrm{OR}_N(y_{1,1}, \ldots, y_{1,N}), \ldots, \mathrm{OR}_N(y_{R,1} \ldots, y_{R,N})).$$

  - Let p be a degree $O(R^{1/2} \cdot N^{1/2})$ polynomial approximating $\mathrm{AND}_R(\mathrm{OR}_N, \ldots, \mathrm{OR}_N)$.
  - Then $p(y_{1,1}, \ldots, y_{1,N}, \ldots, y_{R,1}, \ldots, y_{R,N})$ approximates SURJ, with degree $O(\deg(p) \cdot \log R) = O(R^{1/2} \cdot N^{1/2} \cdot \log R)$.

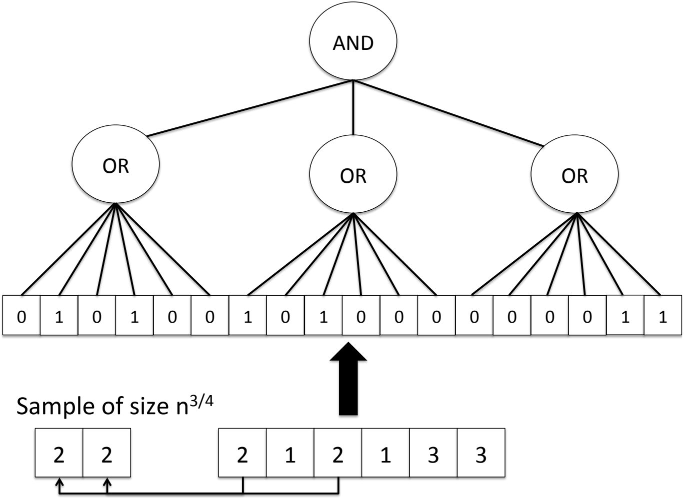
The Upper Bound For SURJ: Second Try

- Fix $R = N/2$. We'll show $\widetilde{\deg}(\mathsf{SURJ}_{R,N}) = \tilde{O}(R^{1/4} \cdot N^{1/2})$.
- We'll want to think of polynomials as computing the probability that a query algorithm outputs 1.
  - E.g., we can think of our "first try" as composing an query algorithm for computing AND_R with R copies of a query algorithm computing OR_N.
- We'll approximate SURJ via a "two-stage" construction.

Stage 1

- Consider a query algorithm that samples $O(n^{3/4})$ inputs.
- Any range item appearing in the sample definitely has frequency at least 1, so we can just "remove it from consideration."
- Stage 2 just needs to determine whether all range items not appearing in the sample have frequency at least 1.
- Let $\mathsf{SURJ}_{\mathsf{unsamp}}$ be the function we need to compute in Stage 2.

Stage 1 Illustrated ($R = 3, N = 6$)



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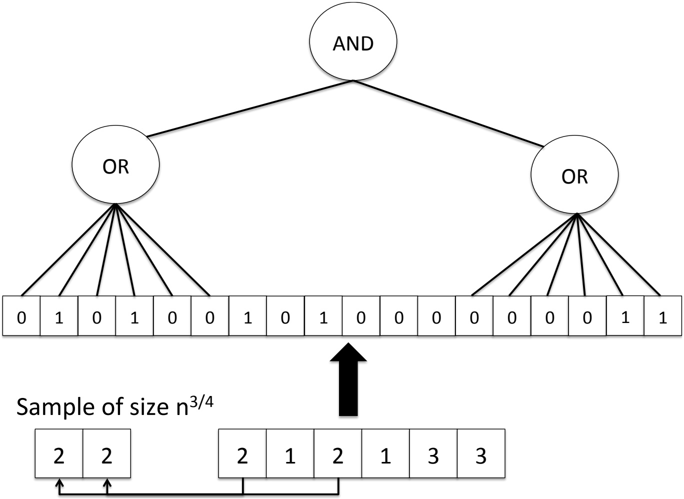

Stage 2

- Key observation: any range item with frequency larger than $T = n^{1/2}$ will appear in the sample at least once, with probability $1 - \exp(-n^{1/4})$.
- i.e., if a range item doesn't appear in the sample, we are <u>really confident</u> that it does not have a very high frequency.

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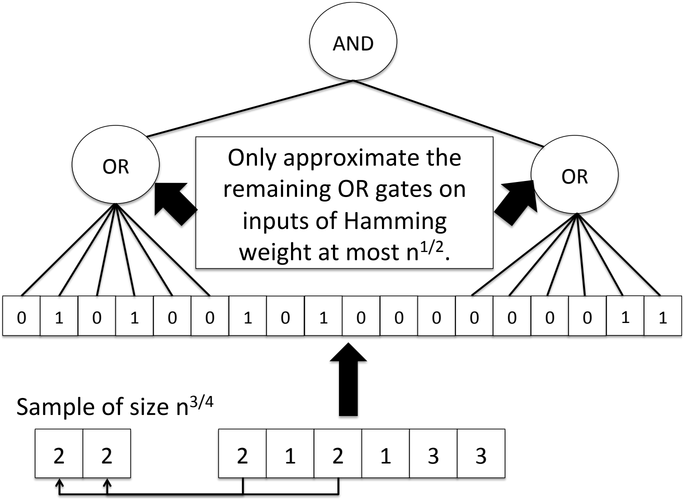
  - If p is fed an input in which some range item has frequency higher than T, then p is allowed to be exponentially large on that input.
  - Specifically, if b unsampled range items have frequency larger than T, then it is okay for $|p(x)|$ to be as large as $\exp(n^{1/4} \cdot b)$.

Stage 2 Illustrated $(R = 3, N = 6)$



Stage 2 Details

Lemma (Chebyshev polynomials)

There is a polynomial q of degree $\tilde{O}(n^{1/4})$ such that

- $|q(x) - \mathrm{OR}_n(x)| \ll 1/n$ *for all* $|x| \leq n^{1/2}$.
- $|q(x)| \leq \exp\left(\tilde{O}(n^{1/4})\right)$ *otherwise.*

Theorem

For $x = (x_1, \ldots, x_R)$, *let* $b(x_1, \ldots, x_R) = \#\{i \colon |x_i| > n^{1/2}\}$. *There is a polynomial q of degree* $\tilde{O}(R^{1/2} \cdot N^{1/4})$ *such that:*

- $|q(x) - \mathrm{AND}_R \circ \mathrm{OR}_N(x)| \leq 1/3$ *if* $b(x) = 0$.
- $|p(x)| \leq \exp\left(\tilde{O}(b(x) \cdot n^{1/4})\right)$ *otherwise.*

Proof.

Let h approximate AND_R, and let $p = h \circ q$. □

Lower Bound Analysis for SURJ

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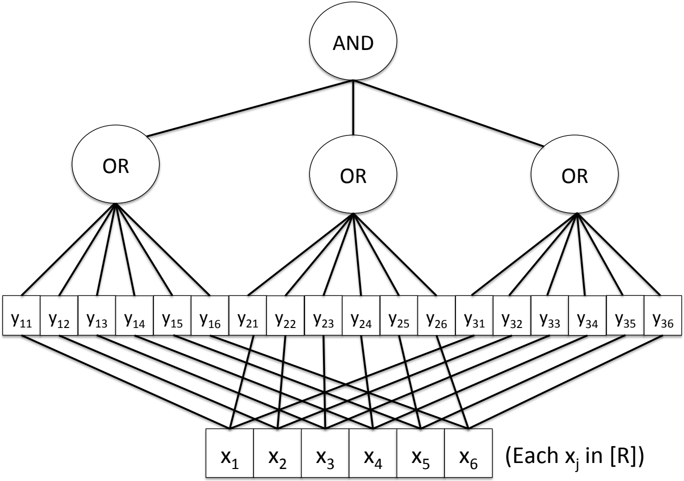
SURJ Illustrated $(R = 3, N = 6)$



(Each x_j in [R])

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    - Follows from a symmetrization argument (Ambainis 2003).
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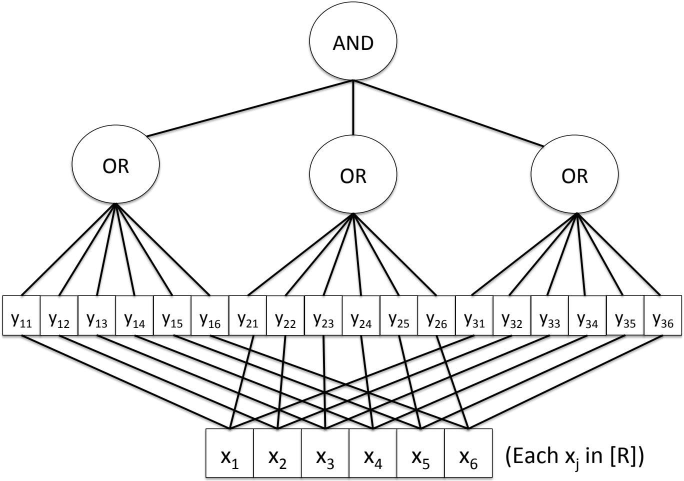

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- Step 2: Prove that for some $N = O(R)$, this promise problem requires degree $\gtrsim \Omega(R^{3/4})$.
  - Builds on the "dual combining technique" used earlier to analyze $\mathrm{AND\text{-}OR}_n$ (with no promise).

Overview of Step 2

Prove That For Some $N = O(R)$, Approximating $\mathrm{AND}_R \circ \mathrm{OR}_N$ Under the Promise That The Input Has Hamming Weight **At Most** N Requires Degree $\gtrsim R^{3/4}$.

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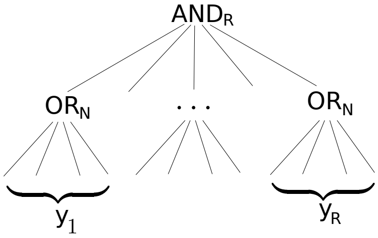

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- Fact (cf. Razborov and Sherstov 2008): Suppose

$$\sum_{|y|>N} |\psi_{\textbf{AND-OR}}(y)| \ll R^{-D}.$$

  - Then we can "post-process" $\psi_{\textbf{AND-OR}}$ to "zero out" any mass it places it inputs of Hamming weight larger than N.
  - While ensuring that the resulting dual witness still has pure high degree $\min\{D, \mathsf{PHD}(\psi_{\textbf{AND-OR}})\}$.

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- $|\psi_{\textbf{AND-OR}}(y_1,\ldots,y_R)|$ resembles product distribution: $\prod_{j=1}^{R} |\psi_{\textbf{OR}}(y_j)|$
- So it is exponentially more biased toward low Hamming weight inputs than $\psi_{\textbf{OR}}$ itself.

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- Also by (2), each $|y_i| = N^{1/4}$ contributes a factor of $1/\mathrm{poly}(N)$.

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- So total mass on these inputs is $\exp(-\Omega(N^{3/4}))$.

General Hardness Amplification Within AC^0

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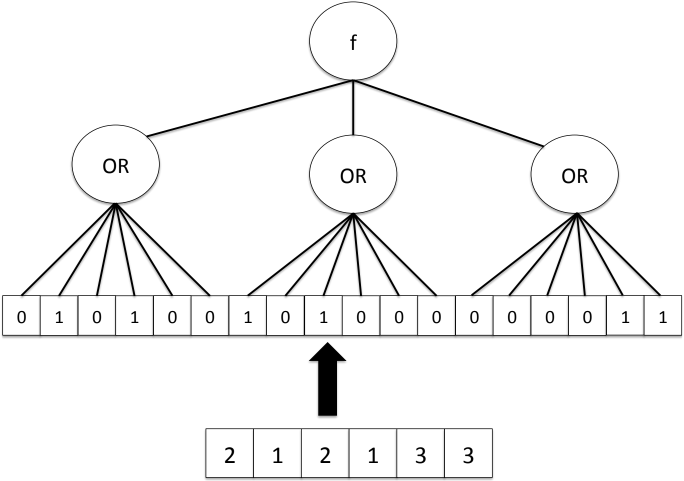
- Recall: When we apply our hardness amplification to $f = \mathrm{AND}_R$, the "harder" function F is precisely SURJ.
- For a general function f, what is the "harder" function F?

First Attempt: Amplifying Hardness of $f: \{-1,1\}^R \to \{-1,1\}$ ($R=3, N=6$)



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- First attempt at handling general f fails when $f = \mathrm{OR}$.
  - $F(x) = \mathrm{OR}_R(\mathrm{OR}_N(y_{1,1}, \ldots, y_{1,N}), \ldots, \mathrm{OR}_N(y_{R,1} \ldots, y_{R,N}))$ has (exact) degree 0.

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- Let $R' = R \log R$. For $f \colon \{-1,1\}^R \to \{-1,1\}$, the real* definition of F is:

$$F(x) = (f \circ \mathrm{AND}_{\log R})(\mathrm{OR}_N(y_{1,1}, \ldots, y_{1,N}), \ldots, \mathrm{OR}_N(y_{R',1}, \ldots, y_{R',N}))$$

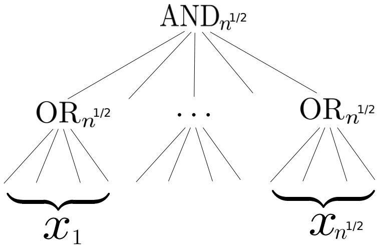
*This is still a slight simplification. Also need to introduce a dummy range item that is ignored by F.

Future Directions

- Resolve the quantum query complexity of k-distinctness, counting triangles, graph collision, etc.
- Prove an $\Omega(n^{k/(k+1)})$ lower bound on approximate degree of the k-sum function?
  - Its quantum query complexity is known to be $\Theta(n^{k/(k+1)})$.
- An $\Omega(n)$ lower bound on the approximate degree of AC^0?
- A sublinear <u>upper bound</u> for DNFs of polynomial size? Or even polynomial size AC^0 circuits?
  - Either result would yield new circuit lower bounds (namely, for $\mathsf{AC}^0 \circ \mathsf{MOD}_2$ circuits).
- Extend our bounds on $\widetilde{\deg}_\epsilon(f)$ from $\epsilon = 1/3$ to ϵ much closer to 1.
  - We believe our techniques can extend to give a $\Omega(n^{1-\delta})$ lower bound on the <u>threshold degree</u> of AC^0.

Thank you!

Analysis of the Dual Witness for the AND-OR Tree



The Combining Method [SZ09, She09, Lee09]

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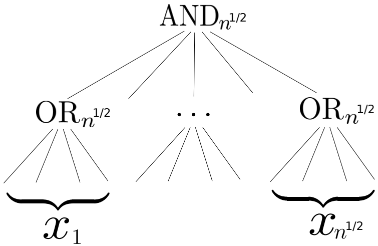
(C chosen to ensure $\psi_{\textbf{AND-OR}}$ has L_1-norm 1).



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Pure High Degree Analysis [She09]

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$$C \cdot \operatorname{sgn}(\psi_{\textbf{IN}}(x_1)) \cdot \operatorname{sgn}(\psi_{\textbf{IN}}(x_2)) \cdot \prod_{i=1}^{3} |\psi_{\textbf{IN}}(x_i)|$$
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$$C \cdot \text{sgn}(\psi_{\textbf{IN}}(x_1)) \cdot \text{sgn}(\psi_{\textbf{IN}}(x_2)) \cdot \prod_{i=1}^{3} |\psi_{\textbf{IN}}(x_i)|$$

$$= \psi_{\textbf{IN}}(x_1) \cdot \psi_{\textbf{IN}}(x_2) \cdot |\psi_{\textbf{IN}}(x_3)|$$

- Each term of $\psi_{\textbf{AND-OR}}$ is the product of PHD($\psi_{\textbf{OUT}}$) polynomials over disjoint variable sets, each of pure high degree at least PHD($\psi_{\textbf{IN}}$).
- So $\psi_{\textbf{AND-OR}}$ has pure high degree at least PHD($\psi_{\textbf{OUT}}$) $\cdot$ PHD($\psi_{\textbf{IN}}$).

(Sub)Goal: Show $\psi_{\textbf{AND-OR}}$ has high correlation with $\mathrm{AND\text{-}OR}$

Correlation Analysis

$$\psi_{\textsf{\textbf{AND-OR}}}(x_1, \ldots, x_{n^{1/2}}) := C \cdot \psi_{\textsf{\textbf{OUT}}}(\ldots, \operatorname{sgn}(\psi_{\textsf{\textbf{IN}}}(x_i)), \ldots) \prod_{i=1}^{n^{1/2}} |\psi_{\textsf{\textbf{IN}}}(x_i)|$$

- Idea: Show

$$\sum_{x \in \{-1,1\}^n} \psi_{\textsf{\textbf{AND-OR}}}(x) \cdot \text{AND-OR}_n(x) \approx \sum_{y \in \{-1,1\}^{n^{1/2}}} \psi_{\textsf{\textbf{OUT}}}(y) \cdot \text{AND}_{n^{1/2}}(y).$$

- Intuition: We are feeding $\operatorname{sgn}(\psi_{\textsf{\textbf{IN}}}(x_i))$ into $\psi_{\textsf{\textbf{OUT}}}$.
- $\psi_{\textsf{\textbf{IN}}}$ is **correlated** with $\text{OR}_{n^{1/2}}$, so $\operatorname{sgn}(\psi_{\textsf{\textbf{IN}}}(x_i))$ is a "decent predictor" of $\text{OR}_{n^{1/2}}$.
- But there are errors. Need to show errors don't "build up".

Correlation Analysis

- Goal: Show

$$\sum_{x \in \{-1,1\}^n} \psi_{\textsf{AND-OR}}(x) \cdot \mathrm{AND\text{-}OR}_n(x) \approx \sum_{y \in \{-1,1\}^{n^{1/2}}} \psi_{\textsf{OUT}}(y) \cdot \mathrm{AND}_{n^{1/2}}(y).$$

- Case 1: Consider any $y = (\operatorname{sgn} \psi_{\textsf{IN}}(x_1), \ldots, \operatorname{sgn} \psi_{\textsf{IN}}(x_{n^{1/2}})) \neq$ **All-True**.
- There is some coordinate of y that equals FALSE. Only need to "trust" this coordinate to force $\mathrm{AND\text{-}OR}_n$ to evaluate to FALSE on $(x_1, \ldots, x_{n^{1/2}})$. So errors do not build up!

Correlation Analysis

- Case 2: Consider $y =$ **All-True**.
- $\mathrm{AND}_{n^{1/2}}(y) = \mathrm{AND\text{-}OR}_n(x_1, \ldots, x_{n^{1/2}})$ only if all coordinates of y are "error-free".
- Fortunately, ψ_{IN} has a special **one-sided error** property:

  If $\mathrm{sgn}(\psi_{\mathsf{IN}}(x_i)) = -1$, then $\mathrm{OR}_{n^{1/2}}(x_i)$ is **guaranteed** to equal -1.

Summary of Correlation Analysis

- Two Cases.
- In first case (feeding at least one FALSE into $\psi_{\mathbf{OUT}}$), errors did not build up, because we only needed to "trust" the FALSE value.
- In second case (all values fed into $\psi_{\mathbf{OUT}}$ are TRUE), we needed to trust all values. But we could do this because $\psi_{\mathbf{IN}}$ had one-sided error.

One-Sided Approximate Degree

- A real polynomial p is a <u>one-sided ϵ-approximation</u> for f if

$$|p(x) - 1| < \epsilon \quad \forall x \in f^{-1}(-1)$$

$$p(x) \geq 1 \quad \forall x \in f^{-1}(1)$$

- $\widetilde{\text{odeg}}_{-,\epsilon}(f) =$ min degree of a one-sided ϵ-approximation for f.
- $\widetilde{\text{odeg}}_{-}(f) := \widetilde{\text{odeg}}_{-,1/3}(f)$ is the one-sided approximate degree of f.

Dual Formulation of $\widetilde{\text{odeg}}_{-}$

Primal LP (Linear in ϵ and coefficients of p):

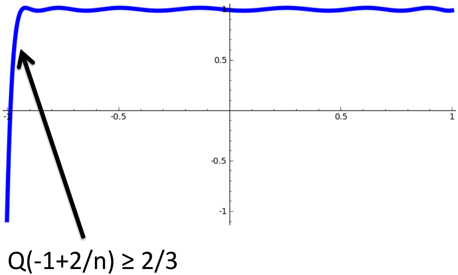
$$\begin{aligned} \min_{p,\epsilon} \quad & \epsilon \\ \text{s.t.} \quad & |p(x) - 1| \leq \epsilon && \text{for all } x \in f^{-1}(-1) \\ & p(x) \geq 1 && \text{for all } x \in f^{-1}(1) \\ & \deg p \leq d \end{aligned}$$

Dual LP:

$$\begin{aligned} \max_{\psi} \quad & \sum_{x \in \{-1,1\}^n} \psi(x) f(x) \\ \text{s.t.} \quad & \sum_{x \in \{-1,1\}^n} |\psi(x)| = 1 \\ & \sum_{x \in \{-1,1\}^n} \psi(x) q(x) = 0 && \text{whenever } \deg q \leq d \\ & \psi(x) \geq 0 \quad \forall x \in f^{-1}(1) \end{aligned}$$

Proof that $\widetilde{\text{odeg}}\_(\text{AND}_n) = \Omega(\sqrt{n})$

We argued that the symmetrization of any $1/3$-approximation to AND_n had to look like this:



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