Semi-Streaming Algorithms for Annotated Graph Streams

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Data Streaming Model

- Stream: m elements from universe of size N
  - e.g., $\langle x_1, x_2, \ldots, x_m \rangle = 3,5,3,7,5,4,8,7,5,4,8,6,3,2, \ldots$

- Goal: Compute a function of stream, e.g., number of distinct elements, frequency moments, heavy hitters.

- Challenge:

  (i) Limited working memory, i.e., polylog(m,N).

  (ii) Sequential access to adversarially ordered data.

  (iii) Process each update quickly.

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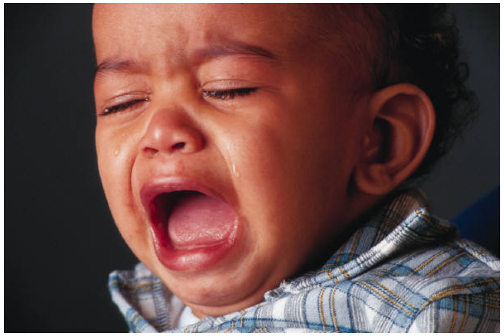
- Bad news: many graph problems cannot be solved (or even approximated) by a streaming algorithm in $o(n^2)$ space.
  - Example: distinguishing graphs with 0 triangles from those with 1 triangle.



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- Bad news: many graph problems cannot be solved (or even approximated) by a streaming algorithm in $o(n^2)$ space.
  - Example: distinguishing graphs with 0 triangles from those with 1 triangle.
- A bright spot: some simple properties can be solved in O(n*polylog(n)) space.
  - Examples: bipartiteness, connectivity
  - These are called **semi-streaming algorithms**.

Outsourcing

- Many applications require outsourcing computation to untrusted service providers.
    - Main motivation: commercial cloud computing services.
    - Also, weak peripheral devices; fast but faulty co-processors.
    - Volunteer Computing (SETI@home, World Community Grid, etc.)

- User requires a guarantee that the cloud performed the computation correctly.

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Model of Streaming Verification for This Work

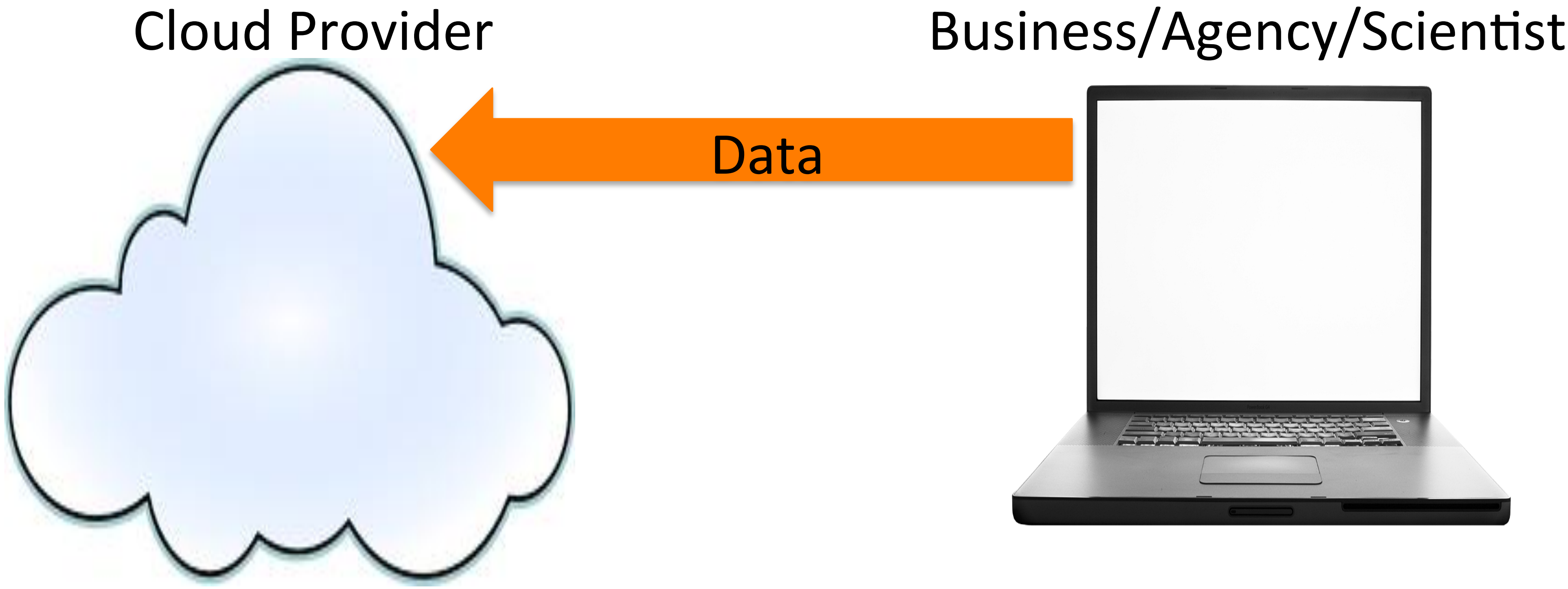
- Chakrabarti et al. [CCM09/CCMT14] introduced the model of **annotated data streams**.
  - One message (non-interactive) model: P and V both observe stream. Afterward, P sends V an email with the answer, and a proof attached.
  - Think of V's streaming pass over the input as occurring while V is uploading data to the cloud.

Annotated Data Streams

Cloud Provider

Business/Agency/Scientist

Annotated Data Streams



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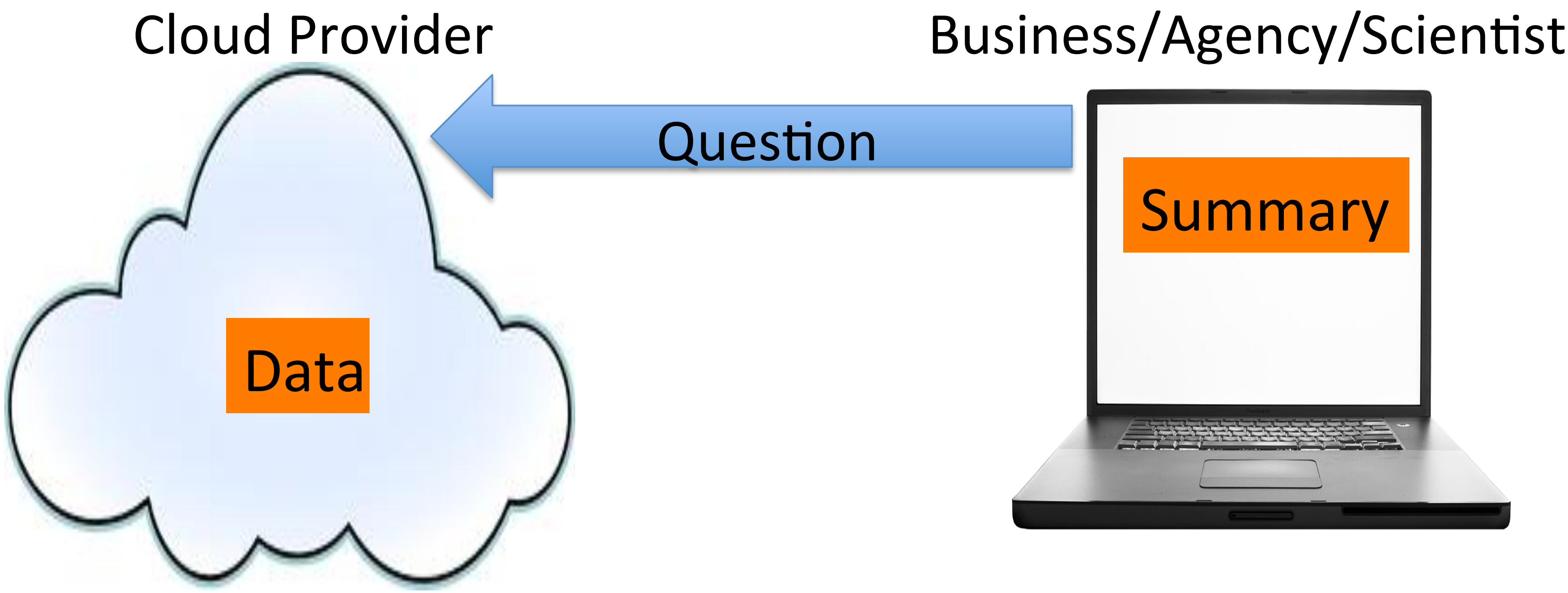

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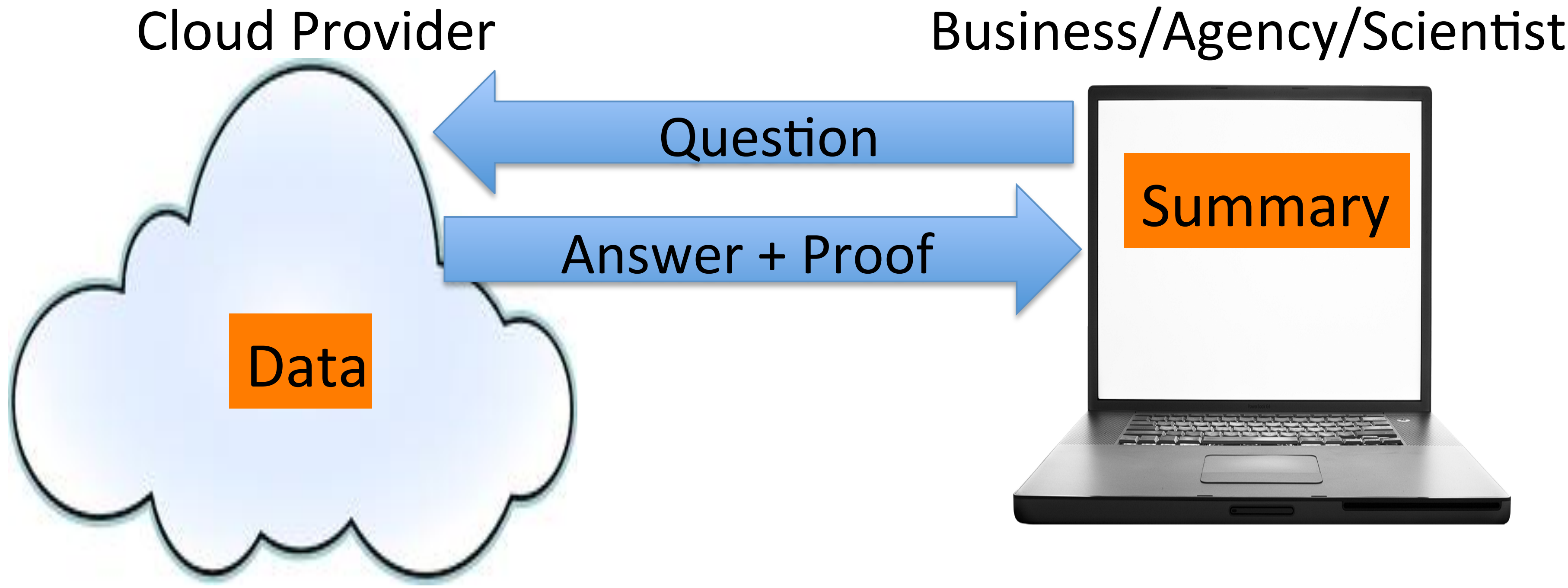

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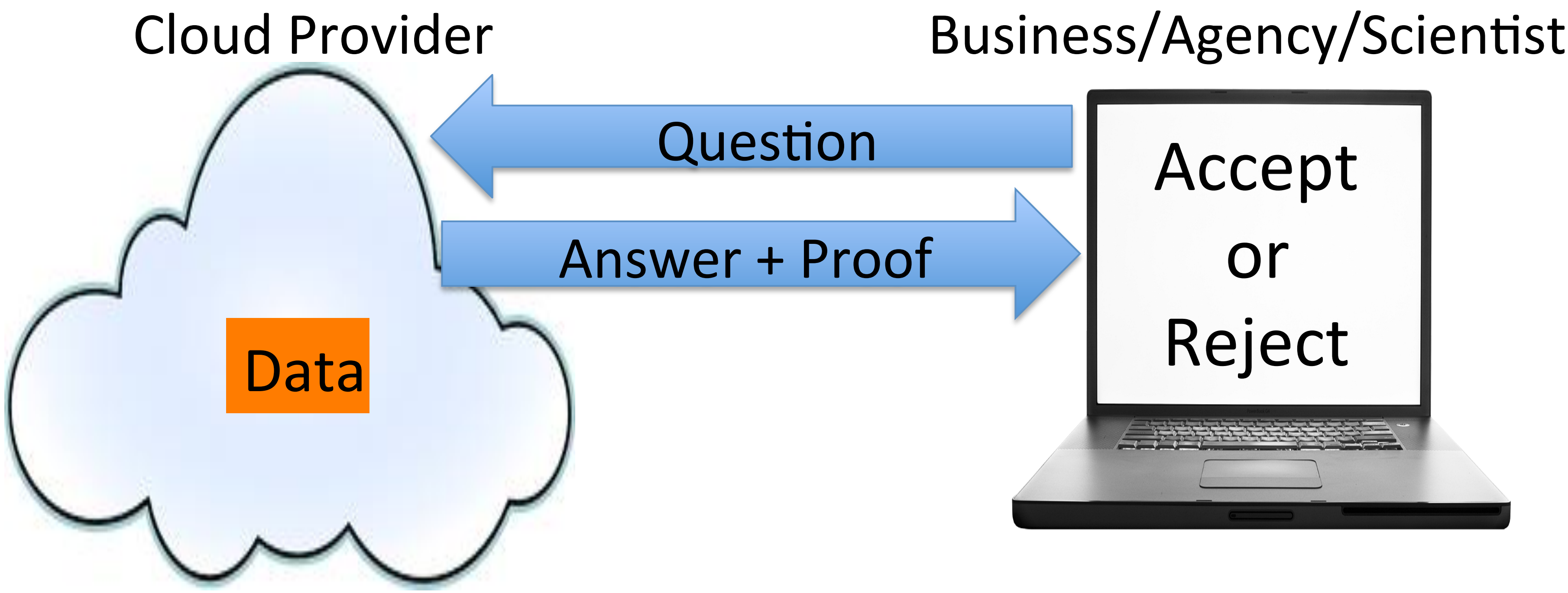


Annotated Data Streams

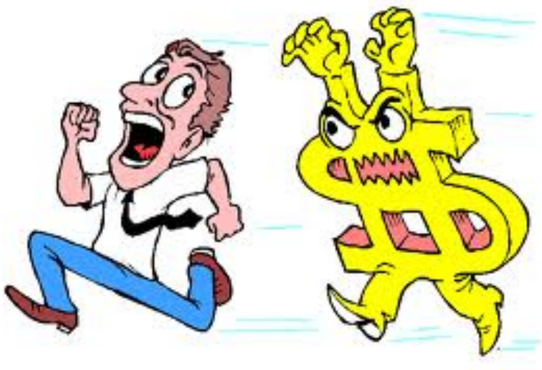
- Prover P and Verifier V observe a stream.

- P solves problem, tells V the answer.
  - P appends a proof that the answer is correct.

- Requirements:
  - 1. Completeness: an honest P can convince V to accept.
  - 2. Soundness: V will catch a lying P with high probability (secure even if P is computationally unbounded).

Costs of Annotated Data Streams

- Two main costs: proof length, and V's working memory. Both must be **sublinear** in input size.



Costs of Annotated Data Streams

- Two main costs: proof length, and V's working memory. Both must be **sublinear** in input size.
  - Notation: an (h,v)-protocol is one with proof length O(h) and memory cost O(v) for V.
  - The **total cost** of the protocol is h+v.
  - For graph problems on n nodes, refer to a protocol of total cost O(n*polylog(n)) as a **semi-streaming scheme.**
- Other costs: running time of both P and V.

Another Model of Streaming Verification

- Cormode et al. [CTY12] introduced more general model called **streaming interactive proofs** (SIPs) that allows multiple rounds of interaction between P and V.
  - Annotated data streams correspond to 1-message SIPs.

Comparison of Two Models

- Pros of multi-round model:
  1. Exponentially reduces space and communication cost. Often (polylog n, polylog n).
- Cons of multi-round model:
  1. P must do significant computation *after each message*.
  2. More coordination needed; network latency might be an issue.

- Pros of single-message model:
  1. Space and communication still reasonable.
  2. P can do all computation at once, just send an email with proof attached.
  3. Reusability: can run the protocol on a stream, then receive more stream updates and seamlessly run the protocol on the updated stream.

History of Annotated Data Streams and SIPs

- [CCM09, CTY12, KP13, GR13, CTY12, PSTY13, CCMTV14, KP14, DTV15, ADDRV16] all study variants of these models.
- [CMT12] gave efficient implementations of protocols from [CCM09, CMT10] (and from the literature on "classical" interactive proofs).

Our Results

- Part 1: We give semi-streaming schemes for **exactly** solving two graph problems in **dynamic** graphs streams that require $\Omega(n^2)$ space in the standard streaming model.
  - Counting triangles.
  - Maximum cardinality matching.
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  - Only known semi-streaming schemes were for bipartite perfect matching, and shortest s-t path in graphs of polylogarithmic diameter [CMT10, CCM09/CCMT14].
- Part 2: We show two graph problems that are **just as hard** in the annotated data streaming model.
  - Connectivity and bipartiteness.
  - Caveat: the result holds in the "XOR edge update" model.

Semi-Streaming Schemes for Counting Triangles

Summary of Annotated Data Streaming Protocols for Counting Triangles

Reference	(Proof Length, Space Cost)	Total Cost Achieved
[CCMT14]	$(n^2, 1)$	$O(n^2)$
[CCMT14]	(h, v): for any $h \cdot v = n^3$	$O(n^{3/2})$
This work	(n, n)	O(n)

- [CCMT14] proved a lower bound that any (h, v) protocol must satisfy $h \cdot v > n^2$.
- Question of whether there is semi-streaming scheme for the problem is Question #47 on sublinear.info (posed by Cormode at Bertinoro 2011).
- Interesting properties of our solution:
  - V's final state depends on the order of the stream.
  - Our approach does not allow smooth tradeoffs of proof length and space cost.

Outline of the Exposition

1. Sum-Check Protocol of [LFKN90]
   (a) Simple, non-interactive variant
   (b) Full Interactive Sum-Check Protocol
2. Low-Degree Extensions
3. A Simple, Interactive Protocol for Counting Triangles, via (b)
4. The Annotated Data Streaming Protocol, via (a).

Sum-Check Protocol [LFKN90], Simplified

- Let **F** be a finite field of (prime) size at least n^3.
- Associate elements of **F** with integers in the natural way.

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- Claim: Suppose we identify a univariate polynomial g (that depends on the input stream) over $\mathbf{F}$ such that
  1. The number of triangles in the graph equals $\sum_{b \in [n]} g(b)$.
  2. For a randomly chosen point $r \in \mathbf{F}$, V can evaluate $g(r)$ using space v with a single streaming pass over the stream.

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  Then there is a $(\deg(g), v)$–protocol for counting triangles.
- Proof: P sends a polynomial s (specified by its coefficients) claimed to equal g. V checks if $s(r) = g(r)$ and if so outputs $\sum_{b \in [n]} s(b)$.
- Completeness is obvious. Soundness error is at most $\deg(g)/|\mathbf{F}|$.

Sum-Check Protocol [LFKN90]

- Suppose the input specifies a d-variate polynomial g over field **F.**
- Goal: compute the quantity:

$$\sum_{b_1 \in [n]} \sum_{b_2 \in [n]} \cdots \sum_{b_d \in [n]} g(b_1, \ldots, b_d)$$

- Costs:
  - d rounds of interaction.
  - Total communication is $O(d \cdot \deg(g))$.
  - Space cost for V is the space to evaluate g at a random point.

Low-Degree Extensions

- Define $E:[n]\times[n]\rightarrow\{0,1\}$ by:

  $E(u,v)=1$ if edge (u,v) appears in G.

  $E(u,v)=0$ otherwise.

- Let $\mathbf{F}$ be a field, and let $\tilde{E}(u,v)$ denote the bivariate polynomial over $\mathbf{F}$ of degree n in each variable that agrees with E at all inputs in $[n]\times[n]$.
- Fact: For any point $(r_1,r_2)\in\mathbf{F}^2$, V can evaluate $\tilde{E}(r_1,r_2)$ in constant space with a single streaming pass over the input.

$$E : [n] \times [n] \rightarrow \{0,1\}$$

0	1
1	1

$$\widetilde{E} : \mathbf{F}^2 \rightarrow \mathbf{F}$$

0	1	2	3	4	5
1	1	1	1	1	1
2	1	0	-1	-2	-3
3	1	-1	-3	-5	-7
4	1	-2	-5	-8	-11
5	1	-3	-7	-11	-15

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A Simple Interactive Protocol for Counting Triangles

- The number of triangles in G equals
  $$\sum_{u\in[n]}\sum_{v\in[n]}\sum_{z\in[n]}\tilde{E}(u,v)\cdot\tilde{E}(v,z)\cdot\tilde{E}(u,z).$$

- Get a 3-round (n, 1)-protocol by applying sum-check to the **trivariate** polynomial $g(X,Y,Z)=\tilde{E}(X,Y)\cdot\tilde{E}(Y,Z)\cdot\tilde{E}(X,Z)$.

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- Can get a 2-round (n, n)-protocol by applying sum-check to the **bivariate** polynomial $g'(X,Y)=\tilde{E}(X,Y)\cdot\sum_{z\in[n]}\tilde{E}(Y,z)\cdot\tilde{E}(X,z)$.
  - V can evaluate g' at a random point $(r_1,r_2)\in\mathbf{F}^2$ in space O(n) by computing $\tilde{E}(r_1,r_2)$, as well as $\tilde{E}(r_1,z)$ and $\tilde{E}(r_2,z)$ for all $z\in[n]$.

The Annotated Data Streaming Protocol: Outline

- To get a semi-streaming scheme, we need to write the number of triangles in the graph as $\sum_{b \in [n]} g(b)$ for a **univariate** polynomial g of degree $O(n)$ that V can evaluate at any point in $O(n)$ space.

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- Key idea: g will itself be a **sum** of polynomials g_i, one for each stream update.
- $\sum_{z \in [n]} g_i(z)$ will count the number of triangles **completed** at time i.
- Hence, the total number of triangles will be

$$\sum_{i \le m} \left(\sum_{z \in [n]} g_i(z) \right) = \sum_{z \in [n]} \left(\sum_{i \le m} g_i(z) \right) = \sum_{z \in [n]} g(z).$$

- Need to ensure each g_i has degree $O(n)$ and that for any r and all i, V can evaluate $g_i(r)$ in $O(n)$ space.

The Annotated Data Streaming Protocol: Details

- Define $E_i : [n] \times [n] \rightarrow \{0,1\}$ by:

  $E_i(u,v) = 1$ if edge (u,v) appears in G after i stream updates.

  $E_i(u,v) = 0$ otherwise.

- If the i'th stream update is edge (u_i, v_i), define

$$g_i(Z) = \tilde{E}_i(u_i, Z) \cdot \tilde{E}_i(v_i, Z).$$

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$$g_i(Z) = \tilde{E}_i(u_i, Z) \cdot \tilde{E}_i(v_i, Z).$$

- Observe:
  - g_i is a univariate polynomial of degree at most 2n.
  - $\sum_{z \in [n]} g_i(z)$ is the number of triangles completed by (u_i, v_i) at time i.
  - V can evaluate $g_i(r) = \tilde{E}_i(u_i, r) \cdot \tilde{E}_i(v_i, r)$ by maintaining $\tilde{E}_i(u,r)$ for all $u \in [n]$ at all times i.
  - Hence, V can also evaluate $g(r) = \sum_{i \le m} g_i(r)$ in $O(n)$ space.

Semi-Streaming Scheme for Maximum Cardinality Matching

Summary of Annotated Data Streaming Protocols for Maximum Cardinality Matching

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[CMT10]	(m, 1)	O(m)
This work	(n, n)	O(n)

- [CCMT14] proved a lower bound that any (h, v) protocol must satisfy $h*v > n^2$ (even in the bipartite case).

Lower Bounds for Connectivity and Bipartiteness

Overview of Lower Bound and Proof

- Claim: In the XOR update model, any annotated data streaming protocol for Connectivity and Bipartiteness must have total cost Ω(n). These problems are solvable in O(n*polylog(n)) space without a prover.

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  - We reduce INDEX on n^2 bits to Connectivity on graphs with n nodes.
  - Reduction is tailored to the annotated data streaming model: P helps V perform the reduction.
  - This is necessary.
    - Connectivity on n nodes is **easier** than INDEX on n^2 bits in the standard streaming model, but they're equally hard in annotated data streaming model.

Open Questions

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- Exhibit any graph problem that **cannot** be solved by a semi-streaming scheme.
- Do there exist non-trivial (i.e., $o(n^2)$ total cost) annotated data streaming protocols for any of the following?
  - Shortest *s*-*t* path in general graphs
  - Graph diameter
  - Computing the value of a maximum flow.
- Do there exist annotated data streaming protocols of $o(n)$ total cost for Connectivity or Bipartiteness in the insert-only update model? The strict turnstile update model?
- Is it possible to give an annotated data streaming protocols for Counting Triangles of space cost $o(n)$ and help cost $o(n^2)$?

Thank you!