Attribute-Efficient Learning and Weight-Degree Tradeoffs for Polynomial Threshold Functions

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Attribute-Efficient Learning

- Attribute-efficient learning is a clean framework capturing the problem of learning in the presence of **irrelevant information**.
  - Especially important in the age of Big Data.

- Consider a scientist trying to identify genetic causes of a disease.
  - The disease depends on the interaction of a small number of genes.
  - The scientist collects a massive amount of genetic data from participants.
  - Only a small amount of this information is actually relevant to the function being learned (the mapping of genes to a subject's phenotype).

Attribute-Efficient Learning

- Goal of an algorithm for attribute-efficient learning:
  - Run in time poly(n), where n is total number of attributes.
  - Use a number of examples which is polynomial in the description length of the function f to be learned.
  - The latter can be substantially smaller than n if most of the attributes are irrelevant.

Comparison to Junta Problem

- The most general version of the problem of problem of learning in the presence of irrelevant information is called the "Junta Problem" [Blum-Langley 1997, Mossel-O'Donnell-Servedio 2004].
  - Assume nothing about f other than that it depends on k << n attributes.
  - Uniform-distribution variant of Junta Problem called "the most important open question in uniform distribution learning" by MOS.

- Our goal is both more and less ambitious than the uniform-distribution Junta Problem.
  - We want to learn under *arbitrary distributions.*
  - But are willing to assume the relevant attributes interact in structured ways.
  - We focus on attribute-efficient learning of *decision lists*.

Decision Lists

- A length k decision list of $x_1, \ldots, x_n$ is a sequence of "if-then-else" statements:

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d] & x_5 \arrow[r] \arrow[d] & x_9 \arrow[r] \arrow[d] & x_3 \arrow[r] \arrow[d] & x_7 \arrow[r] \arrow[d] & 1 \\ 0 & 1 & 1 & 0 & 1 & \end{tikzcd}$$

- Attribute-efficiently learning DLs is a well-studied and challenging open problem.
- First posed by [Blum 1992], subsequently considered by many authors [Blum-Langley 1997, Valiant 1999, Servedio 2000, Nevo-El-Yaniv 2002, Klivans-Servedio 2006, Long-Servedio 2006].
- DLs are PAC-learnable in poly(n) time, but seem to lie on boundary of tractability in the attribute-efficient setting.

Mistake-Bounded Learning

- We establish our results in the *mistake-bounded model*.
  - Standard conversions [Littlestone 1989] turn mistake bounds to sample complexity bounds on PAC learning algorithms.
- Mistake-Bounded model:
  - Learning consists of a sequence of trials. In each trial, the learner is given some x from $\{0,1\}^n$ and outputs h(x), her guess as to what f(x) is.
    - If $h(x) = f(x)$, great!
    - If $h(x) \neq f(x)$, learner is charged a mistake.
- Goal: design an efficient algorithm that minimizes number of mistakes over all possible (infinite) sequences of trials.

Algorithmic Machinery

- Theorem (Expanded-Winnow Algorithm) [Klivans-Servedio 2004]:

Let $f(x)=\mathrm{sgn}(p(x_1, \ldots, x_n))$, where p is a degree-d polynomial with integer coefficients whose absolute values sum to W. Then we can learn f in time $n^{O(d)}$ per example and mistake bound $O(W^2 d \log(n))$.

- p is called a *polynomial threshold function* (PTF) for f, and W is called the *weight* of p.

- Corollary: Attribute-efficient learning of DLs reduces to showing that every length k decision list has a low-degree, low-weight PTF.

What was known?

- Theorem [Klivans-Servedio 2004]: Let f be a length k DL. For every $d \leq k^{1/3}$, there is a degree d, weight $2^{O(k/d^2)}$ PTF computing f.

- Theorem [Beigel 1994]: There is a length k decision list f such that for any $d \leq k$, any degree d PTF computing f requires weight $2^{\Omega(k/d^2)}$.

- So both theorems are tight at low degrees ($d<k^{1/3}$). But it was open what happens at higher degrees.

- We show that at higher degrees, neither theorem is tight!

New Results

- Theorem: Let f be a length k DL. For every $d \geq k^{1/3}$, there is a degree d, weight $2^{O((k/d)^{\wedge}1/2)}$ GPTF* computing f.

  *A GPTF is slightly more expressive than a PTF, and just as useful for learning purposes.

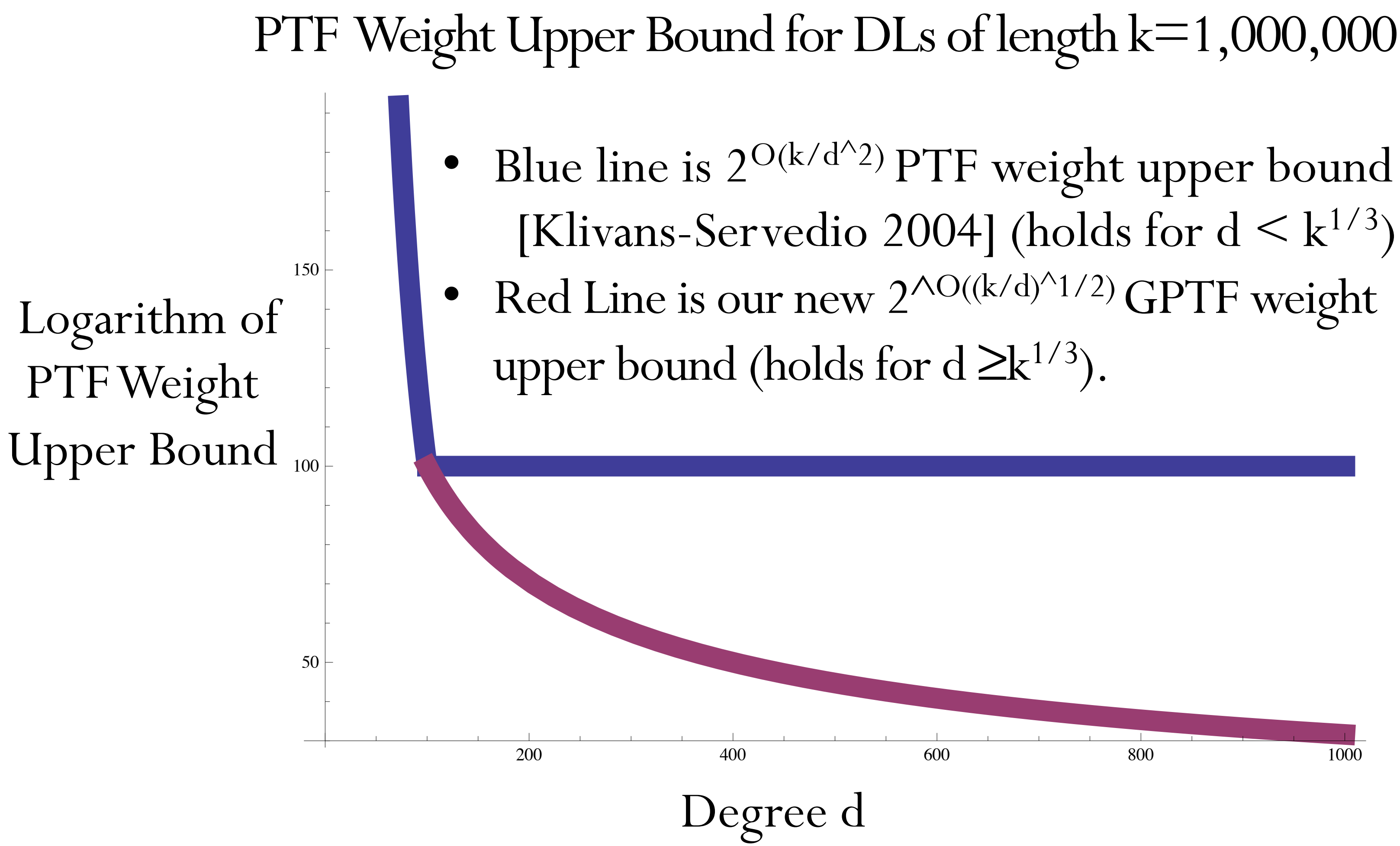
- Theorem: There is a length k DL such that for any $d \leq k$, any degree d PTF computing f requires weight $2^{\Omega((k/d)^{\wedge}1/2)}$.

- Both of these theorems improve on prior work when the degree is relatively high ($d > k^{1/3}$).

- The main remaining gap is that our upper bound uses GPTFs while our lower bound applies only to PTFs.

Comparison of New Upper Bound to Prior Work [Klivans-Servedio 2004]

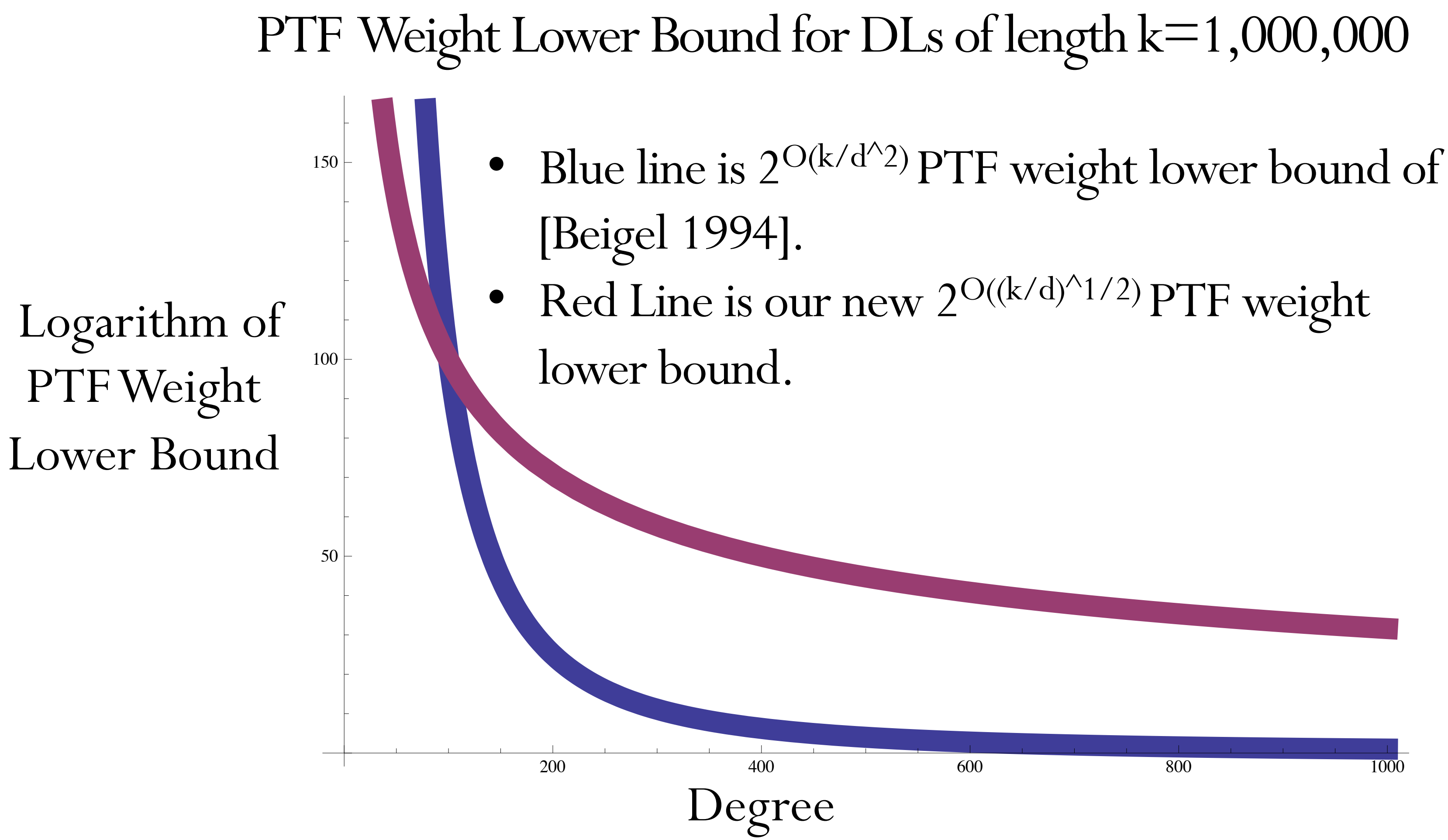
- Blue line is $2^{O(k/d\^2)}$ PTF weight upper bound of [Klivans-Servedio 2004] (holds for $d < k^{1/3}$).
- Red Line is our new $2^{\wedge O((k/d)\^1/2)}$ GPTF weight upper bound (holds for $d \geq k^{1/3}$).



Comparison of Our Algorithm to Prior Work

	run time	mistake bound
Winnow Algorithm [Littlestone 1988]	n	$2^k \log(n)$
Halving Algorithm [Littlestone 1988]	n^k	$k \log(n)$
Klivans-Servedio (for every $d \leq k^{1/3}$)	n^d	$2^{O(k/d^2)} \log(n)$
Servedio-Tan-Thaler (for every $d \geq k^{1/3}$)	n^d	$2^{O((k/d)^{1/2})} \log(n)$

Comparison of New Lower Bound to Prior Work [Beigel 1994]



Upper Bound Proof Sketch

- Given: a length k DL f.
- Break f into k/b "blocks" of length b.
- Closely approximate each block i in the L_∞-norm with a low-degree polynomial $p_i(x)$.
  - If block i "makes a decision", $p_i(x)$ outputs a value close to ± 1.
  - Otherwise, $p_i(x)$ outputs 0.
- Put the approximations together to get a PTF p for the entire decision list f.
  - $p(x) = \sum_i 3^i p_i(x)$.
  - The highest block i to "make a decision" will dominate the output of p, so $f = \text{sgn}(p(x))$.

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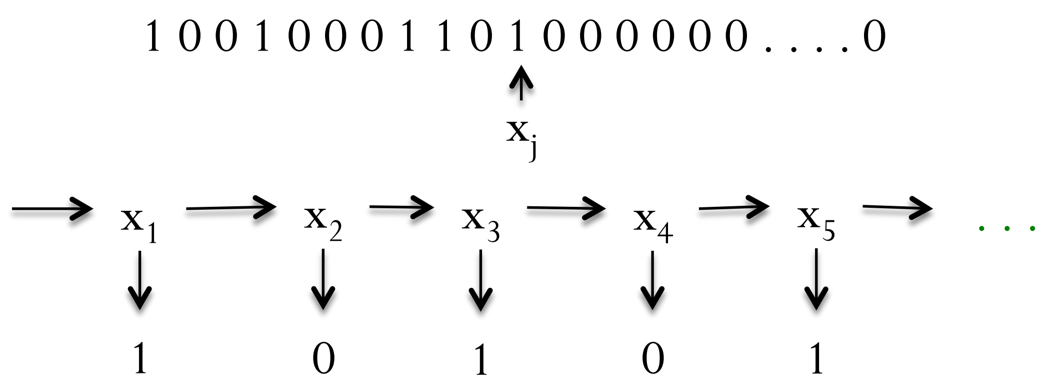
- Put the approximations together to get a PTF p for the entire decision list f.
  - $p(x) = \Sigma_i 3^i p_i(x)$.
  - The highest block i to "make a decision" will dominate the output of p, so $f = sgn(p(x))$.
  - Degree of p equals degree of the p_i's.
  - Weight of p depends on the number of blocks and the weight of the p_i's. Choose block length to balance these contributions.

Upper Bound Proof Sketch

- Klivans-Servedio use degree d Chebyshev polynomials to construct each approximating polynomial $p_i(x)$.
- But when the d is relatively large, the degree d Chebyshev polynomials have very high weight.
  - Instead, we use *lower* degree Chebyshev polynomials, composed with a high-degree monomial.
  - This allows us to achieve lower weight approximating polynomials $p_i(x)$ than those obtained by Klivans-Servedio for the same degree.

Lower Bound Proof Sketch

- We prove a lower bound for a specific decision list, ODD-MAX-BIT (OMB).
- Look at the right-most bit set to 1. If it is at an odd coordinate, output 1, else output 0.



Lower Bound Proof Sketch

- Lower bound argument shows that "block-based" approach of our upper bound is intrinsic.
- Break the OMB function into k/b blocks of length b.
- Show that you can take any PTF p for OMB and turn it into a polynomial q closely approximating each block.
  - q has the same degree and weight as p.
- Beigel used Markov's inequality from approximation theory to conclude that q has to have high degree, and hence p has to have high degree as well.

Lower Bound Proof Sketch

- Markov's inequality bounds the derivative of a polynomial q in terms of its degree.

- We prove a new Markov-type inequality which takes into account *both* the degree of q and the size of its coefficients.

Lower Bound Proof Sketch

- **Markov's Inequality**: Let q : [-1,1] → [-1,1] be a real polynomial with deg(q) ≤ d. Then $\max_{|x|\leq 1} |q'(x)| \leq d^2$.

- **Our Markov-type Inequality**: Let q : [-1,1] → [-1,1] be a real polynomial with deg(q) ≤ d and coefficients of absolute value at most W. If $\frac{1}{2} \leq \max_{|x|\leq 1} |q(x)|$, then

$$\max_{|x|\leq 1} |q'(x)| = O(d*\max\{d, \log(W)\}).$$

- If $W << 2^d$, our inequality is tighter than Markov's.
- This allows us to improve Beigel's lower bound for OMB when d is relatively large.

Lower Bound Proof Sketch

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- **Our Markov-type Inequality**: Let $q : [-1,1] \rightarrow [-1,1]$ be a real polynomial with $\deg(q) \leq d$ and coefficients of absolute value at most W. If $\frac{1}{2} \leq \max_{|x|\leq 1} |q(x)|$, then

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- If $W << 2^d$, our inequality is tighter than Markov's.
- This allows us to improve Beigel's lower bound for OMB when d is relatively large.
- Tight example for Markov: degree d Chebyshev polynomials. Tight example for our inequality degree d Chebyshev polynomials composed with a high-degree monomial.
- Same intuition applied for our upper bound.

Conclusions

- We provide new positive and negative results for attribute-efficient learning of decision lists.
- Our results rely on a careful study of PTF weight-degree tradeoffs for decision lists.
  - Both our upper and lower bounds improve over prior work when the allowed degree of the (G)PTF is relatively high.

- Open questions:
  - Cryptographic hardness of true attribute-efficient learning of length k decision lists? [Servedio 2000] has partial results in this direction.
  - New algorithms: beyond PTFs?
  - Moving beyond DLs: Attribute-efficient learning of more expressive concept classes like decision trees and DNFs?

Thank you!