Streaming Graph Computations with a Helpful Advisor

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Data Streaming Model

- Stream: m elements from universe of size n
  - e.g., $S=<x_1, x_2, \ldots, x_m> = 3,5,3,7,5,4,8,7,5,4,8,6,3,2, \ldots$

- Goal: Compute a function of stream, e.g., median, number of distinct elements, frequency moments, heavy hitters.

- Challenge:

  (i) Limited working memory, i.e., sublinear(n,m).

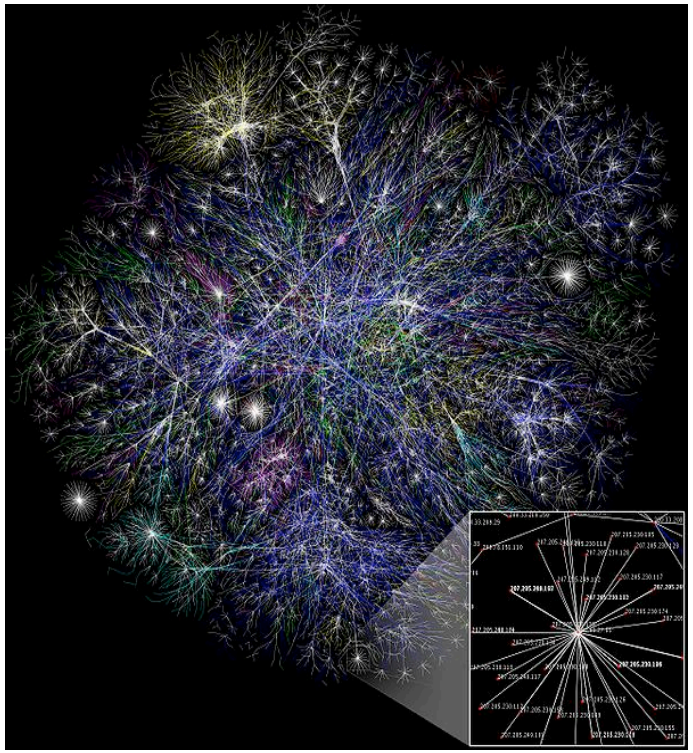
  (ii) Sequential access to adversarially ordered data.

  (iii) Process each update quickly.

Slide derived from [McGregor 10]

Graph Streams

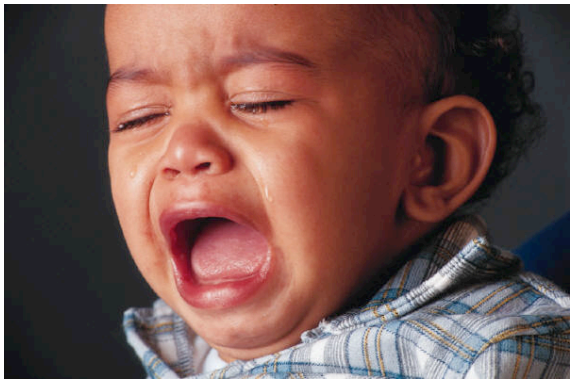
- $S = \langle x_1, x_2, \ldots, x_m \rangle$; $x_i \in [n] \times [n]$
- A defines a graph G on n vertices.
- Goal: compute properties of G.
- Challenge: subject to usual streaming constraints.



Snapshot of Internet Graph
Source: Wikipedia

Bad News

- Many graph problems are impossible in standard streaming model (require linear space or many passes over data).
- E.g. $\Omega(n)$ space needed for connectivity, bipartiteness. $\Omega(n^2)$ space needed for counting triangles, diameter, perfect matching.
- Often hard even to approximate.
- Graph problems ripe for outsourcing.



Outsourcing Models

- Stream Punctuation [Tucker et al. 05], Proof Infused Streams [Li et al. 07], Stream Outsourcing [Yi et al. 08], Best-Order Model [Das Sarma et al. 09] (is a special case of our model)

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- [Chakrabarti et al. 09] Online Annotation Model: Give streaming algorithm access to powerful *helper* H who can annotate the stream.
  - Main motivation: Commercial cloud computing services such as Amazon EC2. Helper is untrusted.
  - Also, Volunteer Computing (SETI@home. Great Internet Mersenne Prime Search, etc.)
  - Weak peripheral devices.

Online Annotation Model

- **Problem**: Given stream S, want to compute f(S):

$S=<x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_m>$

- Helper H: augments stream with h-word annotation:

$(S,a)=<x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_m, a_1, a_2, \ldots, a_h>$

- Verifier V: using v words of space and random string r, run verification algorithm to compute g(S,a,r) such that for all a either:

  a)$Pr_r[g(S,a,r) = f(S)]=1$ (we say a is valid for S) or

  b) $Pr_r[g(S,a,r) = \perp] \geq 1-\delta$ (we say a is δ-invalid for S)

  c) And at least one a is valid for S.

Note: this model differs slightly from [Chakrabarti et al. 09].

Online Annotation Model

- Two costs: words of annotation h and working memory v.
  - We refer to (h, v)-protocols.
  - Primarily interested in minimizing v.
  - But strive for optimal tradeoffs between h and v.
  - Proves more challenging for graph streams than numerical streams. Algebraic structure seems critical.

Fingerprinting

- Need a way to test multiset equality (e.g. to see if two streams have the same frequency distribution).
  - But need to do so in a streaming fashion.
  - We often use this to make sure H is "consistent".
- Solution: fingerprints.
  - Hash functions that can be computed by a streaming verifier.
  - If $S \neq S'$ as frequency distributions, then $f(S) \neq f(S')$ w.h.p.
- We choose a fingerprint function f that is linear. $f(S \circ S') = f(S) + f(S')$ where $\circ$ denotes concatenation. Will need this for matrix-vector multiplication.

Two Approaches To Designing Protocols

1. Prove matching upper and lower bounds on a quantity.
   - One bound often easy: just give feasible solution.
   - Proving optimality more difficult. Usually requires problem structure.

2. Use H to "verify" execution of a non-streaming algorithm.

Streaming LP problem

- Suppose stream A contains (only the non-zero) entries of matrix $\mathbf{A}$, vectors $\mathbf{b}$ and $\mathbf{c}$, interleaved in any order (updates are of the form e.g. "add y to entry (i,j) of $\mathbf{A}$"). The LP streaming problem on A is to determine $\max \{\mathbf{c}^T \mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$.

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  - Protocol ("naïve" matrix-vector multiplication):
    1. H provides primal-feasible solution $\mathbf{x}$.
    2. For each row i of $\mathbf{A}$:

       Repeat entries of $\mathbf{x}$ and row i of $\mathbf{A}$ in order to prove feasibility. Fingerprints ensure consistency.
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- Details on precision of rationals are skipped.

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- Corollary: (m, 1) protocols for max-flow, min-cut, minimum-weight bipartite perfect matching, and shortest *s*-*t* path. Lower bound of $hv = \Omega(n^2)$ for all four.
- **A** is sparse for the problems above, which suits the naïve protocol. For denser **A**, can get optimal tradeoffs between *h* and *v*.

Dense Matrix-Vector Multiplication

- We will get optimal $(n^{1+\alpha}, n^{1-\alpha})$ protocol. Lower bound: $hv = \Omega(n^2)$.
  - Corollary I: Protocols for dense LPs, effective resistances, verifying eigenvalues of Laplacian.

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  - Corollary I: Protocols for dense LPs, effective resistances, verifying eigenvalues of Laplacian.
  - Corollary II: Optimal tradeoffs for Quadratic Programs, Second-Order Cone Programs. (n^2, 1) protocol for Semi-definite Programs.

Dense Matrix-Vector Multiplication

- First idea: Treat as n separate inner-product queries, one for each row of A.
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  - Multiplies *both h* and *v* by n, as compared to a single inner-product query.

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- First idea: Treat as n separate inner-product queries, one for each row of A.
  - Worse than "naïve" solution.
  - Multiplies *both* h and v by n, as compared to a single inner-product query.
- Key observation: one vector, $\mathbf{x}$, in each inner-product query is constant.
  - This plus linear fingerprints lets us just multiply h by n.
  - v will be the same as for a *single* inner product query.

Approach 2: Simulate an Algorithm

- Main tool: Offline memory checker [Blum et al. '94]. Allows efficient verification of a sequence of accesses to a large memory.

- Lets us convert any deterministic algorithm into a protocol in our model.

- Running time of the algorithm in the RAM model becomes annotation size h.

Memory Checker [Blum et al. '94]

- Consider a *memory transcript* of a sequence of reads and writes to memory.
- A transcript is *valid* if each read of address i returns the last value written to that address.
- Memory checker requires transcript be provided in a carefully chosen ("augmented") format.
  - Augmentation blows up transcript size only by constant factor.
- *V* checks validity by keeping a constant number of fingerprints and performing simple local checks on the transcript.

Simulation Theorem

- Any graph algorithm *M* in RAM model requiring time t can be (verifiably) simulated by an (m+t, 1)-protocol.

- *Proof sketch:*
  - Step 1: H first plays a valid adjacency-list representation of *G* to "initialize memory".
  - Step 2: H provides a valid augmented transcript T of the read and write operations performed by algorithm.
  - V checks validity using memory-checker. V also checks all read/write operations are as prescribed by *M*.

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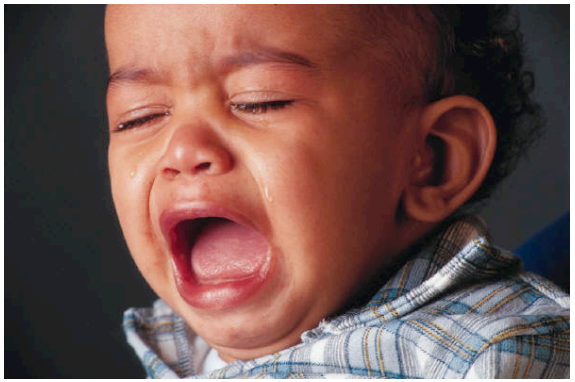
- Lower bounds: hv= Ω (n^2) for single source and all-pairs shortest paths. *hv*= Ω (n^2) for MST if edge weights specified incrementally.

Pitfall of Memory-Checking

Cannot simulate *randomized* algorithms



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- Let A be adjacency matrix of G.
- $(I + A)^l_{ij} > 0$ if and only if there is a path of length at most l from i to j.
- Protocol:

1. H claims diameter is l
2. Use repeated squaring to prove $(I+A)^l$ has an entry that is 0, and $(I+A)^{l+1} \neq 0$ for all i,j).

Summary

- (m, 1)-protocol for max-matching. $hv = \Omega(n^2)$ lower bound for dense graphs, so we can't do better.
- (m, 1)-protocols for LPs TUM IPs. $hv = \Omega(n^2)$ lower bound for several TUM IPs.
- Optimal $(n^{1+\alpha}, n^{1-\alpha})$-protocol for dense matrix-vector multiplication. $(n^{1+\alpha}, n^{1-\alpha})$-protocols for effective resistance, verifying eigenvalues of Laplacian or Adjacency matrix, LPs, QPs, SOCPs.
- General simulation theorem; applications to MST, shortest paths.
- $(n^2 \log n, 1)$ protocol for Diameter. $hv = \Omega(n^2)$ lower bound.

Open questions

- Tradeoffs between h, v for matching, MST, diameter?
- Distributed computation: Protocols that work with Map-Reduce.
- What if we allow multiple rounds of interaction between H and V? Can we get exponentially better protocols?

Thank you!